Logit Dynamics with Concurrent Updates for Local Interaction Games

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joint work with
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Motivating example

Write down a number between 1 and 100.

Your number should be as close as possible to
half of the average
of all numbers we write.

Motivating example

The standard game-theoretic way

- Numbers are at most 100, so the average will be at most 100, and half of the average will be at most 50
- **I will not write a number larger than** 50

Motivating example

The standard game-theoretic way

- Numbers are at most 100, so the average will be at most 100, and half of the average will be at most 50
- I will not write a number larger than 50
- If none writes a number larger than 50, then the average will be at most 50, and half of the average will be at most 25
- **I will not write a number larger than** 25

Motivating example

The standard game-theoretic way

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- If none writes a number larger than 25,...
- ...
- **Prediction: Everyone writes 1!**

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- I will not write a number larger than 25
- If none writes a number larger than 25,. . .
- . . .
- **Prediction: Everyone writes 1!**

Do you believe that prediction?

Motivating example

A previous experiment

STOC poster session at FCRC'11

Half of the average

12.2

Motivating example

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Standard game theoretic assumption

Rationality common knowledge

This is too strong assumption in several cases

- Limited knowledge
- Limited computational power
- Limited rationality

Randomized best-response

Nash equilibria = Steady states of best-response dynamics

Idea

Relaxation of best-response dynamics

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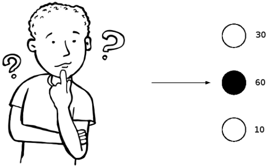

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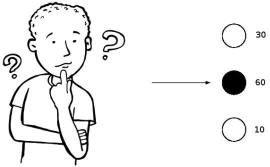

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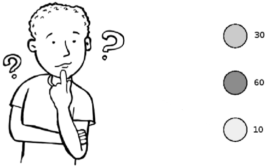

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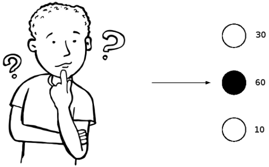

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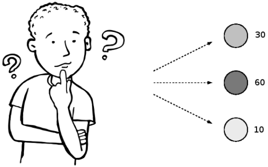

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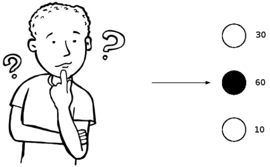

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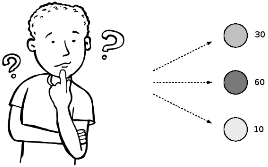

Logit Choice Function [McFadden, 1974]

From profile $\mathbf{x} = (x_1, \ldots, x_n)$ player i chooses strategy y with probability proportional to $e^{\beta u_i(\mathbf{x}_{-i}, y)}$.

Randomized best-response

Logit choice function

$$p_i(y \mid \mathbf{x}) \sim e^{\beta u_i(\mathbf{x}_{-i}, y)}$$

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β = "**Rationality level**"

- $\beta = 0$ players play uniformly at random
- $\beta \to \infty$ players best-respond

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Logit dynamics [Blume, GEB'93]

- Revision process: **choose one player u.a.r.**
- Update rule: **logit choice function**

Previous works on logit dynamics

- **Economics**
  [Blume, GEB'93]:
  Equilibrium selection when $\beta \to \infty$

  [Alós-Ferrer and Netzer, GEB'10]:
  Characterization of stochastically stable states
- **Computer Science**
  [Montanari and Saberi, FOCS'09]:
  Hitting time of the best Nash equilibrium

  [Asadpour, Saberi, WINE'09]:
  Hitting time of the *neighborhood* of best Nash equilibria for Atomic Selfish Routing and Load Balancing.
- **Statistical Mechanics**
  Logit dynamics vs Glauber dynamics

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   Analysis of **mixing time** of logit dynamics for some classes of games [Auletta et al, SPAA'11]

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3. When the time to reach equilibrium is long, can we say something about what happens in the meanwhile?
   Metastability of logit dynamics [Auletta et al, SODA'12]

In this paper

Logit choice function $(p_i(y \mid \mathbf{x}) \sim e^{\beta u_i(\mathbf{x}_{-i}, y)})$ ✓

Revision process (pick one single player at random) ✗

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What happens when all players play simultaneously?

- All-logit ergodic (unique stationary distribution and convergence)

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What happens when all players play simultaneously?

- All-logit ergodic (unique stationary distribution and convergence)
- How do **stationary distribution** for all-logit differ from stationary distribution for one-logit?
- Are there any meaningful **invariant quantities** (that are the same for the one-logit and the all-logit)?

Stationary distribution

Reversibility

What is the stationary distribution for the all-logit dynamics?

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Stationary distribution is **easy** for **reversible** chains.

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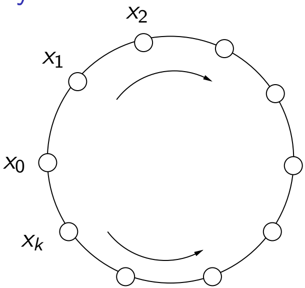
Kolmogorov criterion for reversibility

P is reversible if and only if for every cycle $(x_0, x_1, \dots, x_k, x_0)$

$$P(x_0, x_1)P(x_1, x_2) \cdots P(x_k, x_0)$$

$$=$$

$$P(x_0, x_k)P(x_k, x_{k-1}) \cdots P(x_1, x_0)$$



All-logit dynamics

Potential games and local interaction games

One-logit reversibility

Theorem (Blume, GEB'93)

One-logit *for game $\mathcal{G}$ is* ***reversible*** *if and only if $\mathcal{G}$ is a* ***potential game****.*

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Theorem

All-logit *for game* $\mathcal{G}$ *is* ***reversible*** *if and only is* $\mathcal{G}$ *is a* ***local interaction game****.*

Local interaction games

Potential Games

$\mathcal{G} = ([n], \mathcal{S}, \mathcal{U})$. $\Phi : S_1 \times \cdots \times S_n \to \mathbb{R}$ **exact potential** if for every profile $\mathbf{x}$, for every player i, and for every action y

$$u_i(\mathbf{x}_{-i}, y) - u_i(\mathbf{x}) = -\left[\Phi(\mathbf{x}_{-i}, y) - \Phi(\mathbf{x})\right]$$

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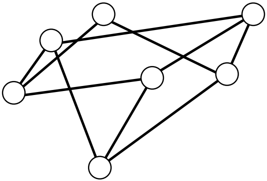
Local interaction games

- **Players are nodes of a graph**



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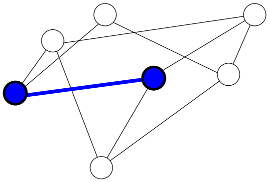
Local interaction games

- Players are nodes of a graph
- **Edges are two-player potential games**



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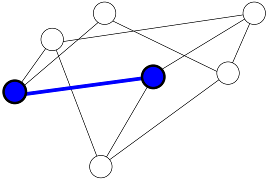

Observation

A local interaction game is a potential game.

All-logit dynamics and local interaction games

Idea of proof

Theorem

Logit dynamics for game $\mathcal{G}$ is reversible if and only if $\mathcal{G}$ is a local interaction game.

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1. **All logit for $\mathcal{G}$ reversible implies $\mathcal{G}$ potential game**
   [It follows from Monderer and Shapley characterization of potential games]

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Idea of proof.

1. All logit for $\mathcal{G}$ reversible implies $\mathcal{G}$ potential game
   [It follows from Monderer and Shapley characterization of potential games]
2. **All-logit for a potential game $\mathcal{G}$ reversible if and only if for every pair of profiles $\mathbf{x}, \mathbf{y}$**

$$K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x}) \quad (1)$$

   where $K(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \Phi(\mathbf{x}_{-i}, y_i) - (n-2)\Phi(\mathbf{x})$
   [It follows from the Kolmogorov criterion for reversibility applied to the all-logit for a potential game]

All-logit dynamics and local interaction games

Stationary

3. **Show that $K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x})$ if and only if $\mathcal{G}$ is a local interaction game**
   [A potential function satisfies $K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x})$ if and only if it is a sum of 2-player potential functions]

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Stationary distribution

$\mathcal{G}$ local interaction game

$$\pi_{\text{all}}(\mathbf{x}) \sim \sum_{\mathbf{y} \in S} e^{-\beta K(\mathbf{x},\mathbf{y})}$$

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All-logit dynamics and local interaction games

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For the **one-logit** it is $\pi_{\text{one}}(\mathbf{x}) \sim e^{-\beta \Phi(\mathbf{x})}$

Observables

Example

$F : \{\text{strategy profiles}\} \to \mathbb{R}$

Question

- Local interaction game $\mathcal{G}$
- π_{one}, π_{all} stationary distributions of one-logit and all-logit

Is there any meaningful observable F such that $\mathbf{E}_{\pi_{\text{one}}}[F] = \mathbf{E}_{\pi_{\text{all}}}[F]$.

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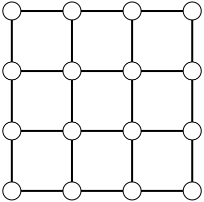
Example (Ising model)



± 1

- $\Phi(\mathbf{x}) = -\sum_{\{i,j\} \in E} x_i x_j$ (Energy)
- $F(\mathbf{x}) = \sum_{i=1}^{n} x_i$ (Magnetization)

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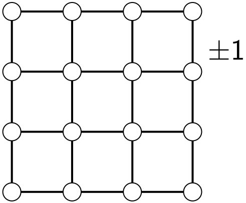

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$$\mathbf{E}_{\pi_{\text{one}}}[F] = \mathbf{E}_{\pi_{\text{all}}}[F]$$

Observables

Decompositions

Local interaction game $\mathcal{G}$, potential Φ, set of strategy profiles S

Decomposition

A permutation $\sigma = (\sigma_1, \sigma_2)$ of $S \times S$ such that for every pair of profiles

- $\sigma_1(\mathbf{x}, \mathbf{y}) = \sigma_2(\mathbf{y}, \mathbf{x})$
- $K(\mathbf{x}, \mathbf{y}) = \Phi(\sigma_1(\mathbf{x}, \mathbf{y})) + \Phi(\sigma_2(\mathbf{x}, \mathbf{y}))$

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Lemma

*$\mathcal{G}$ local interaction game on a **bipartite graph** then $\mathcal{G}$ admits a decomposition.*

Observables

Decompositions

Decomposable observables

Observable F decomposable if decomposition σ exists such that for all $\mathbf{x}, \mathbf{y}$

$$F(\mathbf{x}) + F(\mathbf{y}) = F(\sigma_1(\mathbf{x}, \mathbf{y})) + F(\sigma_2(\mathbf{x}, \mathbf{y}))$$

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Theorem

If F is a decomposable observable then

$$\mathbf{E}_{\pi_{one}}[F] = \mathbf{E}_{\pi_{all}}[F]$$

Conclusions and future works

1. **Logit choice function as a model for players with limited rationality**
2. **Stationary distribution depends on the revision process**

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- Other invariant observables
- **Other revision processes**: Players selected according to some distribution

Thank you!