The Algorithmic Structure of Group Strategyproof Budget-Balanced Cost-Sharing Mechanisms*

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Abstract. We study mechanisms for *cooperative* cost-sharing games satisfying: *voluntary participation* (i.e., no user is forced to pay more her valuation of the service), *consumer sovereignty* (i.e, every user can get the service if her valuation is large enough), *no positive transfer* (i.e., no user receives money from the mechanism), *budget balance* (i.e., the total amount of money that users pay is equal to the cost of servicing them), and *group strategyproofness* (i.e., the mechanism is resistant to coalitions).

We show that mechanisms satisfying all these requirements must obey certain *algorithmic properties* (which basically specify how the serviced users are selected). Our results yield a *characterization of upper continuous mechanisms* (this class is interesting as all known general techniques yield mechanisms of this type). Finally, we extend some of our negative results and obtain the first negative results on the existence of mechanisms satisfying all requirements above. We apply these results to an interesting generalization of cost-sharing games in which the mechanism cannot service certain "forbidden" subsets of users. These *generalized cost-sharing games* correspond to natural variants of known cost-sharing games and have interesting practical applications (e.g., sharing the cost of multicast transmissions which cannot be encrypted).

1 Introduction

Consider a set U of n users that wish to buy a certain service from some service providing company P. Each user $i \in U$ valuates the service offered an amount equal to v_i. This value represents how much user i would benefit from being serviced. Alternatively, v_i quantifies the maximum amount of money that user i is willing to pay for getting the service. The service provider must then develop a so called *mechanism*, that is, a policy for deciding (i) which users should be serviced and (ii) the price that each of them should pay for getting the service.

Mechanisms are complex auctions where users are asked to report their willingness to pay which, in the end, determines the mechanism outcome (i.e., the

* Work supported by the European Project FP6-15964, Algorithmic Principles for Building Efficient Overlay Computers (AEOLUS).

B. Durand and W. Thomas (Eds.): STACS 2006, LNCS 3884, pp. 337–348, 2006.

serviced users and the prices). In particular, the value v_i is known to user i but not to the provider. Hence, users may act *selfishly* and misreport v_i (e.g., trying to get the service for a better price). *Group strategyproof* mechanisms are "resistent" to coalitions of selfish users (see below for a formal definition), and thus particularly appealing for *cost-sharing games* requiring some "reasonable" share of the costs among the (possibly selfish) users.

An instance of a *cost-sharing games* is a pair $\mathcal{I} = (U, C)$, where U is a set of n *users*, and the *cost function* $C : 2^U \to \mathbf{R}^+ \cup \{0\}$ gives the cost $C(Q) > 0$ of servicing all users in a non-empty set $Q \subseteq U$. Each user is a *selfish agent* reporting some bid value b_i (possibly different from v_i); the true value v_i is *privately known* to agent i. Based on the *reported* values $\mathbf{b} = (b_1, \ldots, b_n)$ a *mechanism* $M = (\mathsf{A}, P)$ uses an algorithm A to select a subset $\mathsf{A}(\mathbf{b}|\mathcal{I}) \in 2^U$ of users to service. Moreover, according to the payment functions $P = (P^1, \ldots, P^n)$, each user $i \in \mathsf{A}(\mathbf{b}|\mathcal{I})$ must pay $P^i(\mathbf{b}|\mathcal{I})$ for getting the service. (Users that do not get serviced do not pay.) Hence, the *utility* of agent i when she reports b_i, and the other agents report $\mathbf{b}_{-i} := (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$, is equal to

$$u_i^M(b_i, \mathbf{b}_{-i}|\mathcal{I}) := \begin{cases} v_i - P^i(b_i, \mathbf{b}_{-i}|\mathcal{I}) & \text{if } i \in \mathsf{A}(b_i, \mathbf{b}_{-i}|\mathcal{I}), \\ 0 & \text{otherwise.} \end{cases}$$

Developing economically viable cost-sharing *mechanisms* is a central problem in (cooperative) game theory. In particular, there is a number of natural constraints/goals that, for every instance $\mathcal{I} = (U, C)$ and for every $\mathbf{v} = (v_1, \ldots, v_n)$, a mechanism $M = (\mathsf{A}, P)$ should satisfy/meet: [1]

1. **α-Approximate Budget Balance (α-BB).** The prices charged to all users should recover the cost of servicing them and, at the same time, should not be more than $\alpha > 1$ times this cost. In particular, we require that
$$C(\mathsf{A}(\mathbf{b})) \leq \sum_{i \in \mathsf{A}(\mathbf{b}|\mathcal{I})} P^i(\mathbf{b}|\mathcal{I}) \leq \alpha \cdot C(\mathsf{A}(\mathbf{b})). \tag{1}$$
The lower bound guarantees that there is no loss for the provider. The upper bound implies that a competitor could offer a better price to all users only if coming up with payments such that the above condition is satisfied for some $1 \leq \alpha' < \alpha$. Ideally, one wishes the **budget-balance (BB)** condition, that is, the case $\alpha = 1$. In this case, no competitor can offer better prices to all users in $\mathsf{A}(\mathbf{b})$ without running into a loss. (The cost $C(Q)$ is "common" to all providers and represents the "minimum" cost for servicing Q.)
2. **No Positive Transfer (NPT).** No user receives money from the mechanism, i.e., $P^i(\cdot) \geq 0$.
3. **Voluntary Participation (VP).** We never charge a user an amount of money greater than her *reported* valuation, that is, $\forall b_i, \forall \mathbf{b}_{-i}$ it holds that $b_i \geq P^i(b_i, \mathbf{b}_{-i}|\mathcal{I})$. In particular, a user has always the option of not paying for a service for which she is not interested. Moreover, $P^i(\mathbf{b}|\mathcal{I}) = 0$, for all $i \notin \mathsf{A}(\mathbf{b}|\mathcal{I})$, i.e., only the users getting the service will pay.

[1] Notice that we need to consider all possible $\mathbf{v} = (v_1, \ldots, v_n)$ since the mechanism does not known these values.

4. **Consumer Sovereignty (CS).** Every user is guaranteed to get the service if she reports a high enough valuation, that is, $\forall\ \mathbf{b}_{-i},\ \exists\ \overline{b}_i = \overline{b}_i(\mathbf{b}_{-i})$ such that $i \in \mathsf{A}(\overline{b}_i, \mathbf{b}_{-i}|\mathcal{I})$.
5. **Group Strategyproofness (GSP).** We require that a user $i \in U$ that misreport her valuation (i.e., $b_i \neq v_i$) cannot improve her utility nor improve the utility of other users without worsening her own utility (otherwise, a coalition C containing i would secede). Consider a coalition $C \subseteq U$ of users. For any two vectors $\mathbf{x}$ and $\mathbf{y}$ of length n, $(\mathbf{x}_C, \mathbf{y}_{-C})$ denotes the vector $\mathbf{z} = (z_1, \ldots, z_n)$ such that $z_i = x_i$ if $i \in C$ and $z_i = y_i$ if $i \notin C$. The group strategyproofness requires that if the inequality
$$u_i^M(\mathbf{b}_C, \mathbf{v}_{-C}) \geq u_i^M(\mathbf{v}_C, \mathbf{v}_{-C}) \tag{2}$$
holds for all $i \in C$ then it must hold with equality for all $i \in C$ as well.
Notice that, since we require the condition on Eq. 2 to hold for every $\mathbf{v} = (\mathbf{v}_C, \mathbf{v}_{-C})$, replacing $\mathbf{b}_{-C}$ by $\mathbf{v}_{-C}$ does not change the definition of group strategyproofness. Hence, the special case of $C = \{i\}$ yields the weaker notion of *strategyproofness*: $\forall b_i$ and $\forall \mathbf{b}_{-i}$ it holds that
$$u_i^M(v_i, \mathbf{b}_{-i}) \geq u_i^M(b_i, \mathbf{b}_{-i}), \tag{3}$$
for every user i.

Mechanisms satisfying all requirements above have been deeply investigated (see e.g. [10,9,3]). All known techniques yield mechanisms which select the final set $Q = \mathsf{A}(\mathbf{b})$ among a "sufficiently reach" family $\mathcal{P}^{\mathsf{A}} \subseteq 2^U$ of candidates. More specifically, an invariant of known mechanisms is that one can always find an order $i_1, \ldots, i_n$ of the users such that *each* of the following subsets is given in output for some bid vector $\mathbf{b}$:
$$\underbrace{\{i_1, \ldots, i_n\}}_{Q_1=U}, \underbrace{\{i_2, \ldots, i_n\}}_{Q_2}, \ldots, \underbrace{\{i_j, \ldots, i_n\}}_{Q_j}, \ldots, \emptyset. \tag{4}$$

In general, an algorithm A may consider all possible subsets of U, that is, $\mathcal{P}^{\mathsf{A}} = 2^U$, meaning that every $Q \subseteq U$ is returned for some bid vector $\mathbf{b}$. In some cases, however, it may be convenient/necessary to *never* output certain subsets. There are (at least) two main reasons for this:

1. *Computational complexity.* Computing $C(Q)$ may be NP-hard for certain $Q \subseteq U$. In this case, it may be good to avoid $Q = \mathsf{A}(\mathbf{b})$ since otherwise $M = (\mathsf{A}, P)$ will not run in polynomial time, or it will only guarantee α-BB condition, for some $\alpha > 1$, unless $\mathsf{P}{=}\mathsf{NP}$.
2. *Generalized cost-sharing games.* In many practical applications, certain subsets $Q \subseteq U$ may be "forbidden" in the sense that it is impossible to service all and only those users in Q. We model these applications by introducing *generalized cost-sharing games* where instances are triples $\mathcal{I} = (U, \mathcal{P}, C)$, with $\mathcal{P} \subseteq 2^U$ and $C : \mathcal{P} \to \mathbf{R}^+ \cup \{0\}$. The set $\mathcal{P}$ contains the non-forbidden sets and thus we require $\mathsf{A}(\mathbf{b}) \in \mathcal{P}$, for all $\mathbf{b}$. (We assume $\emptyset \in \mathcal{P}$ and $C(Q) > 0$ for $Q \neq \emptyset$.)

As mentioned above, all known techniques yield mechanisms which are *sequential*, that is, there exists $\sigma = (i_1, \ldots, i_n)$ such that $\mathcal{P}^\sigma \subseteq \mathcal{P}^{\mathsf{A}}$, where $\mathcal{P}^\sigma$ consists of all the subsets listed in Eq. 4 (see Def. 1). This poses severe limitations on which (generalized) cost-sharing games these techniques can "solve efficiently": (i) Polynomial running time can be achieved only if $C(\cdot)$ can be approximated in polynomial time within a factor α for all sets in $\mathcal{P}^\sigma$; (ii) For generalized cost-sharing games, the instance $\mathcal{I} = (U, \mathcal{P}, C)$ must satisfy $\mathcal{P}^\sigma \subseteq \mathcal{P}$. It is then natural to ask whether there exist mechanisms of a totally different type (i.e., *not* sequential) which are more powerful, that is, they are computationally more efficient and/or solve more (generalized) cost-sharing games.

In this work we prove that, for the natural class of *upper continuous* mechanisms [2] (see also Def. 2), the answer to this question is "no". And it remains "no" even if we allow the α-BB condition for an *arbitrarily large* $\alpha < \infty$ (e.g., $\alpha = n$). More specifically, for every upper continuous mechanism $M = (\mathsf{A}, P)$ which is α-BB, VP, CS, NPT and GSP, it must be the case that A is sequential, for *every* $\alpha \geq 1$. Our proofs show an interesting phenomenon: for upper continuous mechanisms satisfying all but the α-BB condition above, the fact that A is not sequential creates a "gap" in the payments which either must be all 0 or cannot be bounded from above (i.e., for every $\beta > 0$ there exists $\mathbf{b}$ such that $P^i(\mathbf{b}) > \beta$).

Our result, combined with a simple upper continuous mechanism given in [14,2], shows that *sequential algorithms characterize upper continuous mechanisms.* This implies that generalized cost-sharing games admits such upper continuous mechanisms if and only if they admit sequential algorithms (see Corollary 5). In particular, relaxing BB to α-BB, for *any* $\alpha > 1$, would not allow for solving a wider class of problems; and the "simple" technique in [14,2] is not less powerful than more complex ones which yield upper continuous mechanisms.

Given our characterization, we can better understand which are the limitations of upper continuous mechanisms satisfying α-BB, NPT, VP, CS and GSP:

1. Polynomial-time mechanisms exist only if $C(\cdot)$ is approximable within polynomial time over $\mathcal{P}^\sigma$, for some σ. If we require BB, then $C(\cdot)$ must be polynomial-time computable over $\mathcal{P}^\sigma$, for some σ.
2. For generalized cost-sharing games, these mechanisms exist only for those instances $\mathcal{I} = (U, \mathcal{P}, C)$ satisfying $\mathcal{P}^\sigma \subseteq \mathcal{P}$, for some σ. Moreover, the factor α in the α-BB condition is totally irrelevant: if α-BB is possible then BB is possible too, for any $\alpha > 1$.

We stress that these are the first lower bounds on (upper continuous) mechanisms satisfying α-BB, NPT, VP, CS and GSP. On one hand, one cannot derive any lower bound on polynomial-time mechanisms from the computational complexity of approximating $C(\cdot)$: indeed, there exists cost-sharing games which admit (upper continuous) polynomial-time BB mechanisms satisfying NPT, VP, CS and GSP [14,15], while the cost function $C(\cdot)$ is NP-hard to approximate within some $\alpha > 1$. On the other hand, generalized cost-sharing games have not been investigated before, though many practical applications require them (see Sect. 4).

We also obtain *necessary* conditions for *general* (i.e., non upper continuous) mechanisms. We use these conditions (Def.s 3 and 4) to prove general lower bounds and that, for two users, upper continuous mechanisms are not less powerful than general ones (basically, every mechanism must be sequential – Corollary 3). We describe several applications of generalized cost-sharing games and of our results in Sect. 4.

Due to lack of space, some of the proofs are omitted; these proofs are available in the full version of the paper [13].

Related Work. Probably the simplest BB, NPT, VP, CS and GSP mechanism is the one independently described in [14,2]: Starting from U, drop users in some fixed order $\sigma = (i_1, \ldots, i_n)$, until some user i_r accepts to pay for the total cost of the current set, that is, $b_{i_r} \geq C(\{i_r, \ldots, i_n\})$.

More sophisticated mechanisms were already known from the seminal works by Moulin and Shenker [10,9]. Their mechanisms employ so called *cross-monotonic cost-sharing methods* which essentially divide the cost $C(Q)$ among all users in Q so that user i would not pay more if the mechanism expands the set Q to some $Q' \supset Q$. Cross-monotonic functions do not exists for several games of interest, thus requiring relaxing BB to α-BB, for some factor $\alpha > 1$ [8,2,1,7,5]. Moreover, cross-monotonicity is difficult to obtain in general (e.g., the works [3,12,6,1,5] derive these cost-sharing methods from the execution of non-trivial primal-dual algorithms).

In [14] the authors prove that Moulin and Shenker mechanisms also work for a wider class of cost-sharing methods termed *self cross-monotonic*. The simple mechanism described above is one of such mechanisms [14]. Also the polynomial-time mechanisms for the Steiner tree game in [14,15] are in this class.

Basically, all known mechanisms are upper continuous, except for the one in [2] which, however, requires $C(\cdot)$ being subadditive. All mechanisms in the literature are either variants of Moulin and Shenker mechanisms [10,9], or have been presented in [2]. In all cases, the mechanisms are sequential (and apart from those in [14,15], they use algorithms such that $\mathcal{P}^{\mathsf{A}} = 2^U$).

Characterizations of BB, NPT, VP, CS and GSP mechanisms are known only for the following two cases: (i) the cost function $C(\cdot)$ is submodular [10,9], or (ii) the mechanism is upper continuous and with no free riders [2] [2]. In both cases, these mechanisms are characterized by cross-monotonic cost-sharing methods.

1.1 Preliminaries and Basic Results

Throughout the paper we let $\mathsf{A}_i(\mathbf{b}) = 1$ if $i \in \mathsf{A}(\mathbf{b})$, and $\mathsf{A}_i(\mathbf{b}) = 0$ otherwise, for all i and all $\mathbf{b}$.

Definition 1. *For any ordering $\sigma = (i_1, \ldots, i_n)$ of the users, we let*

$$\mathcal{P}^\sigma := \{\emptyset\} \cup \{i_j, i_{j+1}, \ldots, i_n\}_{1 \leq j \leq n}.$$

[2] Mechanisms without free riders guarantee that all users in $\mathsf{A}(\mathbf{b})$ pay something.

An algorithm A *is* sequential *if there exists* σ *such that* $\mathcal{P}^\sigma \subseteq \mathcal{P}^\mathsf{A}$*. An instance* $\mathcal{I} = (U, \mathcal{P}, C)$ *of a generalized cost-sharing game* admits a sequential algorithm *if* $\mathcal{P}^\sigma \subseteq \mathcal{P}$*, for some* σ*. A generalized cost-sharing game* admits a sequential algorithm *if every instance of the game does.*

Theorem 1 ([14,2]). *For any ordering* σ *of the users, there exists an upper continuous BB, NPT, VP, CS and GSP mechanism* $M = (\mathsf{A}, P)$ *such that* $\mathcal{P}^\mathsf{A} = \mathcal{P}^\sigma$*. Hence, every instance of a generalized cost-sharing game which admits a sequential algorithm, admits an upper continuous BB, NPT, VP, CS and GSP mechanism.*

The following lemma is a well-known result in mechanism design. (See also [13] for a proof.)

Lemma 1 ([16,11]). *For any strategyproof mechanism* $M = (\mathsf{A}, P)$ *the following conditions must hold:*

$$\mathsf{A}_i(b_i, \mathbf{b}_{-i}) = 1 \Rightarrow \forall b_i' > b_i,\ \mathsf{A}_i(b_i', \mathbf{b}_{-i}) = 1; \tag{5}$$

$$\mathsf{A}_i(b_i, \mathbf{b}_{-i}) = \mathsf{A}_i(b_i', \mathbf{b}_{-i}) \Rightarrow\ P^i(b_i, \mathbf{b}_{-i}) = P^i(b_i', \mathbf{b}_{-i}). \tag{6}$$

Lemma 1 and the CS condition imply that, for every i and every $\mathbf{b}_{-i}$, there exists a threshold $\theta_i(\mathbf{b}_{-i})$ such that agent i is serviced for all $b_i > \theta(\mathbf{b}_{-i})$, while for $b_i < \theta_i(\mathbf{b}_{-i})$ agent i is not serviced. The following kind of mechanism breaks ties, i.e. the case $b_i = \theta_i(\mathbf{b}_{-i})$, in a fixed manner:

Definition 2 (upper continuous mechanisms [2]). *A mechanism* $M = (\mathsf{A}, P)$ *is* upper-continuous *if* $\mathsf{A}_i(x, \mathbf{b}_{-i}) = 1$ *for all* $x \geq \theta_i(\mathbf{b}_{-i})$*, where* $\theta_i(\mathbf{b}_{-i}) := \inf\{y |\ \mathsf{A}_i(y, \mathbf{b}_{-i}) = 1\}$ *(This value exists unless the CS condition is violated.)*

We will use the following technical lemma to show that, if payments are bounded from above, then once a user i bids a "very high" b_i, then this user will have to be serviced no matter what the other agents report.

Lemma 2. *Let* $M = (\mathsf{A}, P)$ *be a strategyproof mechanism satisfying NPT, CS and* $\sum_{i \in U} P^i(\mathbf{b}) \leq \beta$*, for all* $\mathbf{b}$*. Then, there exists* $B = B(\beta) \geq 0$ *such that, for all* i *and all* $\mathbf{b}_{-i}$*,* $\mathsf{A}_i(B, \mathbf{b}_{-i}) = 1$*.*

2 Cost-Sharing Mechanisms and Strategyproofness

2.1 Two Necessary Conditions

We next show that strategyproof α-BB, NPT, VP, CS mechanisms must be able to service (i) all users and (ii) exactly one out of any pair $i, j \in U$. (Of course, for some bid vector $\mathbf{b}$.)

Definition 3. *An algorithm* A *satisfies the* full coverage *property if* $U \in \mathcal{P}^\mathsf{A}$*, that is, the algorithm decides to service all users for some bid vector* $\mathbf{b}$*.*

We next show that full coverage is a necessary condition for obtaining strategyproof mechanisms satisfying NPT and CS and whose prices are bounded from above (a necessary condition for α-BB).

Theorem 2. *If* A *does not satisfy the full coverage property, then any strategyproof mechanism* $M = (\mathsf{A}, P)$ *satisfying NPT and CS will run in an unbounded surplus, that is, for every* $\beta > 0$*, there exists* $\mathbf{b}$ *such that* $\sum_{i\in U} P^i(\mathbf{b}) > \beta$.

Proof. We prove the contraposition. Suppose $\sum_{i\in U} P^i(\mathbf{b}) \leq \beta$, for all $\mathbf{b}$. Then, Lemma 2 implies that $\mathsf{A}(\mathbf{B}) = U$, for some constant $B \geq 0$ and for $\mathbf{B} = (B, \ldots, B)$.

Theorem 2 states that, if the mechanism is not able to service all users, then an unbounded surplus must be created. The result we will prove next is a sort of "dual": if the mechanism is not able to selectively service two users, then it will not collect any money.

Definition 4. *An algorithm* A *satisfies the* weak separation *property if, for any* $i, j \in U$*, the algorithm can return a feasible solution to service only one of them, that is, there exists* $Q \in \mathcal{P}^{\mathsf{A}}$ *such that* $|Q \cap \{i, j\}| = 1$.

Condition weak separation is also necessary for strategyproof mechanisms:

Theorem 3. *If* A *does not satisfy the weak separation condition, then any strategyproof mechanism* $M = (\mathsf{A}, P)$ *satisfying NPT, VP and CS will not collect any money from the users when they report some bid vector* $\mathbf{b}$. *Moreover, mechanism* M *will service a subset* $Q \neq \emptyset$*, thus implying that, mechanism* M *cannot be* α*-BB, for any* $\alpha > 1$.

Proof. Since A does not satisfy the weak separation condition there exist $j, k \in U$ such that

$$\forall \mathbf{b},\ \mathsf{A}_j(\mathbf{b}) = \mathsf{A}_k(\mathbf{b}). \tag{7}$$

Let $(x, \mathbf{0}_{-l})$ denote the vector having the l-th component equal to x and all others being equal 0. Consider the following three bid vectors:

$$\begin{aligned}
\mathbf{b}^{(j)} := (\overline{b}_j, \mathbf{0}_{-j}) = &\ (0, \ldots, 0, \overline{b}_j, 0, \ldots, 0, 0, 0, \ldots, 0)\\
\mathbf{b}^{(k)} := (\overline{b}_k, \mathbf{0}_{-k}) = &\ (0, \ldots, 0, 0, 0, \ldots, 0, \overline{b}_k, 0, \ldots, 0)\\
\mathbf{b}^{(j,k)} := &\ (0, \ldots, 0, \overline{b}_j, 0, \ldots, 0, \overline{b}_k, 0, \ldots, 0)
\end{aligned}$$

with $\overline{b}_j$ and $\overline{b}_k$ such that $\mathsf{A}_j(\mathbf{b}^{(j)}) = 1$ and $\mathsf{A}_k(\mathbf{b}^{(k)}) = 1$. (These two values exist by the CS condition.) Then, Eq. 7 implies $\mathsf{A}_j(\mathbf{b}^{(k)}) = 1$ and $\mathsf{A}_k(\mathbf{b}^{(j)}) = 1$. The CS and NPT conditions imply that $P^j(\mathbf{b}^{(k)}) = 0$ and $P^k(\mathbf{b}^{(j)}) = 0$. We apply Lemma 1 and obtain the following implications:

$$\begin{aligned}
\mathsf{A}_j(\mathbf{b}^{(k)}) = 1 &\Rightarrow \mathsf{A}_j(\mathbf{b}^{(j,k)}) = 1 && (8)\\
&\Rightarrow P^j(\mathbf{b}^{(j,k)}) = P^j(\mathbf{b}^{(k)}) = 0, && (9)\\
\mathsf{A}_k(\mathbf{b}^{(j)}) = 1 &\Rightarrow \mathsf{A}_k(\mathbf{b}^{(j,k)}) = 1 && (10)\\
&\Rightarrow P^k(\mathbf{b}^{(j,k)}) = P^k(\mathbf{b}^{(j)}) = 0. && (11)
\end{aligned}$$

The VP condition, Eq. 9, and Eq. 11 imply that, $P^i(\mathbf{b}^{(j,k)}) = 0$, for $1 \leq i \leq n$. Taking $\mathbf{b} = \mathbf{b}^{(j,k)}$, we get the first part of the theorem. Moreover, Eq.s 8 and 10 prove the second part.

The following result follows from Theorems 2 and 3:

Corollary 1. *Any α-BB strategyproof mechanism $M = (\mathsf{A}, P)$, also satisfying NPT, VP and CS, must use an algorithm A satisfying both the full coverage and weak separation properties.*

The above result implies the first lower bound on *polynomial-time* α-BB, NPT, VP, CS, and GSP mechanisms:

Corollary 2. *If $C(U)$ is NP-hard to approximate within a factor $\alpha \geq 1$, then no α-BB mechanism satisfying NPT, VP, CS and GSP can run in polynomial-time.*

2.2 Characterization of Mechanisms for Two Users

We will prove that, for the case of two users, full coverage and weak separation suffice for the existence of mechanisms. The following fact will be the key property:

Fact 4. *Any algorithm A satisfying the full coverage and weak separation conditions is a sequential algorithm for the case of two users. (Indeed, $U \in \mathcal{P}^{\mathsf{A}}$ and $\{1\} \in \mathcal{P}^{\mathsf{A}}$ or $\{2\} \in \mathcal{P}^{\mathsf{A}}$.)*

The above fact and Corollary 1 imply the following:

Corollary 3. *For generalized cost-sharing games involving two users, the following are equivalent:*

1. *There exists a strategyproof α-BB mechanism $M_\alpha = (\mathsf{A}_\alpha, P_\alpha)$ satisfying NPT, VP and CS;*
2. *Every instance $\mathcal{I} = (U, \mathcal{P}, C)$ admits a sequential algorithm A;*
3. *There exists a group strategyproof BB mechanism $M = (\mathsf{A}, P)$ satisfying NPT, VP and CS.*

Our next result, whose proof is given in the full version of this work [13], shows that Corollary 3 does not apply to the case of three (or more) users.

Theorem 5. *There exists an instance $\mathcal{I} = (U, \mathcal{P}, C)$, with $|U| = 3$, which does not admit sequential algorithms. However, there exist a strategyproof mechanism $M = (\mathsf{A}, P)$ satisfying BB, NPT, VP and CS for this instance. Hence, A is not sequential.*

We stress that the mechanism of the above theorem is not upper continuous nor GSP.

3 Characterization of Upper Continuous Mechanisms

We begin with the following technical lemma.

Lemma 3. *Let $M = (\mathsf{A}, P)$ be a GSP mechanism satisfying NPT and VP. For all $\mathbf{b}' = (b'_i, \mathbf{b}_{-i})$ and $\mathbf{b}'' = (b''_i, \mathbf{b}_{-i})$ such that $\mathsf{A}_i(\mathbf{b}') = \mathsf{A}_i(\mathbf{b}'')$, the following holds for all $j \in U$: if $\mathsf{A}_j(\mathbf{b}') = \mathsf{A}_j(\mathbf{b}'')$ then $P^j(\mathbf{b}') = P^j(\mathbf{b}'')$.*

In this section we will consider bid vectors which take values 0 or some "sufficiently large" $B \geq 0$. Recall that a user bidding B is serviced no matter the other agents bids.

Definition 5. *For any mechanism $M = (\mathsf{A}, P)$ such that $\sum_{i \in U} P^i(\mathbf{b}) \leq \beta$, for all $\mathbf{b}$, we let $B = B(\beta)$ be the constant of Lemma 2, and $\mathcal{P}^{\mathsf{A}}_{\beta} := \{\mathsf{A}(\mathbf{b}) | \mathbf{b} \in \{0, B\}^n\} \subseteq \mathcal{P}^{\mathsf{A}}$. Moreover, $\boldsymbol{\beta}_Q$ denotes the vector whose i-th component is equal to B for $i \in Q$, and 0 otherwise.*

Lemma 4. *Let $M = (\mathsf{A}, P)$ be an upper continuous GSP mechanism satisfying NPT, VP and CS. Moreover, let $\sum_{i \in U} P^i(\mathbf{b}) \leq \beta$, for all $\mathbf{b}$. Then, for all $Q \in \mathcal{P}^{\mathsf{A}}_{\beta}$, it holds that $Q = \mathsf{A}(\boldsymbol{\beta}_Q)$.*

Theorem 6. *Let $M = (\mathsf{A}, P)$ be an upper continuous α-BB mechanism satisfying GSP, NPT, VP and CS. Then, for every $Q \in \mathcal{P}^{\mathsf{A}}_{\beta}$, there exists $i_Q \in Q$ such that $Q \setminus \{i_Q\} \in \mathcal{P}^{\mathsf{A}}_{\beta}$.*

Proof. Notice that α-BB implies $\sum_{i \in U} P^i(\mathbf{b}) \leq \beta$ for all bid vectors $\mathbf{b}$ with $\beta = \max_{Q \in \mathcal{P}^{\mathsf{A}}} \alpha C(Q)$. Thus from Lemma 4, we can assume $Q = \mathsf{A}(\boldsymbol{\beta}_Q)$. First of all, we claim that there is at least one user $i_Q \in Q$ such that $i_Q \notin \mathsf{A}(\boldsymbol{\beta}_{Q \setminus \{i_Q\}})$. Indeed, Lemma 1 implies that, for all $i \in \mathsf{A}(\boldsymbol{\beta}_{Q \setminus \{i\}})$, it must be the case that $P^i(\boldsymbol{\beta}_{Q \setminus \{i\}}) = 0$. Hence, if such an i_Q does not exist, then $\sum_{i \in Q} P^i(\mathbf{b}) = 0$, which contradicts the α-BB condition (i.e. $\sum_{i \in U} P^i(\mathbf{b}) > 0$ for all bid vectors $\mathbf{b}$).

Let us then consider $i_Q \in Q$ such that $i_Q \notin \mathsf{A}(\boldsymbol{\beta}_{Q \setminus \{i_Q\}})$, and let $R := Q \setminus \{i_Q\}$. Lemma 2 implies $R \subseteq \mathsf{A}(\boldsymbol{\beta}_R)$. By contradiction, assume $R \subset \mathsf{A}(\boldsymbol{\beta}_R)$ and let $k \in \mathsf{A}(\boldsymbol{\beta}_R) \setminus R$. We will show that a coalition $C = \{i_Q, k\}$ will violate the GSP condition. To this end, consider the following bid vectors which differ only in the i_Q-th coordinate. We let $* \in \{0, B\}$ denote the coordinates of these two vectors other than i_Q and k:

$$\mathbf{b}^{(1)} = \boldsymbol{\beta}_Q = (*, \ldots, *, B, *, \ldots, *, 0, *, \ldots, *) \tag{12}$$

$$\mathbf{b}^{(2)} = \boldsymbol{\beta}_R = (*, \ldots, *, 0, *, \ldots, *, 0, *, \ldots, *) \tag{13}$$

Since $k \notin R$ and $k \neq i_Q$, it must be the case $k \notin Q = R \cup \{i_Q\} = \mathsf{A}(\mathbf{b}^{(1)})$. From the fact that M is upper continuous, we can choose b_k such that $0 < b_k < \theta_k(\mathbf{b}^{(1)}_{-k})$. Let $b_{i_Q} = P^{i_Q}(\mathbf{b}^{(1)})$ and consider the following bid vector which differs from $\mathbf{b}^{(1)}$ only in the i_Q-th and k-th entries:

$$\mathbf{b}^{(3)} = (*, \ldots, *, b_{i_Q}, *, \ldots, *, b_k, *, \ldots, *).$$

The proof of the following fact is given in the full version [13].

Fact 7. *In the sequel we will use the fact that* $u^M_{i_Q}(\mathbf{b}^{(3)}) = u^M_{i_Q}(\mathbf{b}^{(1)})$ *and* $u^M_k(\mathbf{b}^{(3)}) = u^M_k(\mathbf{b}^{(1)})$, *for* $v_k = b_k$.

We are now ready to show that, under the hypothesis $k \in \mathsf{A}(\mathbf{b}^{(2)})$, the coalition $C = \{i_Q, k\}$ violates the GSP condition. Indeed, consider $\mathbf{v}_C = (v_{i_Q}, v_k) = (b_{i_Q}, b_k)$, $\mathbf{b}_C = (0,0)$ and $\mathbf{v}_{-C} = \mathbf{b}^{(1)}_{-C} = \mathbf{b}^{(2)}_{-C} = \mathbf{b}^{(3)}_{-C}$. Hence, $(\mathbf{v}_C, \mathbf{v}_{-C}) = \mathbf{b}^{(3)}$ and $(\mathbf{b}_C, \mathbf{v}_{-C}) = \mathbf{b}^{(2)}$. Fact 7 implies

$$u^M_{i_Q}(\mathbf{v}_C, \mathbf{v}_{-C}) = u^M_{i_Q}(\mathbf{b}^{(3)}) = u^M_{i_Q}(\mathbf{b}^{(1)}) = v_{i_Q} - P^{i_Q}(\mathbf{b}^{(1)}) = 0 = u^M_{i_Q}(\mathbf{b}^{(2)}),$$

where the last inequality is due to the definition of $\mathbf{b}^{(2)}$ and to the VP condition. (Observe that it must hold $P^{i_Q}(\mathbf{b}^{(2)}) = 0$.) Similarly,

$$u^M_k(\mathbf{v}_C, \mathbf{v}_{-C}) = u^M_k(\mathbf{b}^{(3)}) = u^M_k(\mathbf{b}^{(1)}) = 0 < v_k = u^M_{i_Q}(\mathbf{b}^{(2)}),$$

where the last equality follows from the definition of $\mathbf{b}^{(2)}$, from the VP condition, and from $k \in \mathsf{A}(\mathbf{b}^{(2)})$. The above two inequalities thus imply that the coalition $C = \{i_Q, k\}$ violates the GSP condition. Hence a contradiction derived from the assumption $R \subset \mathsf{A}(\boldsymbol{\beta}_R)$. It must then hold $Q \setminus \{i_Q\} = R = \mathsf{A}(\boldsymbol{\beta}_R) = \mathsf{A}(\boldsymbol{\beta}_{Q\setminus\{i_Q\}})$.

Corollary 4. *If* $M = (\mathsf{A}, P)$ *is an upper-continuous mechanism satisfying* α*-BB, NPT, VP, CS and GSP, then* A *must be sequential.*

Proof. Let $Q_1 := U$ and observe that, from the proof of Theorem 2, $Q_1 = U \in \mathcal{P}^{\mathsf{A}}_\beta$. We proceed inductively and apply Theorem 6 so to prove that $Q_j \in \mathcal{P}^{\mathsf{A}}_\beta$ and therefore we can define $i_j := i_{Q_j}$ such that $Q_{j+1} := Q_j \setminus \{i_j\} \in \mathcal{P}^{\mathsf{A}}_\beta$.

Corollary 5. *For generalized cost-sharing games involving any number of users, the following are equivalent:*

1. *There exists an upper-continuous mechanism* $M_\alpha = (\mathsf{A}_\alpha, P_\alpha)$ *satisfying* α*-BB, NPT, VP, CS and GSP;*
2. *Every instance* $\mathcal{I} = (U, \mathcal{P}, C)$ *admits a sequential algorithm* A*;*
3. *There exists an upper-continuous mechanism* $M = (\mathsf{A}, P)$ *satisfying BB, NPT, VP, CS and GSP.*

4 Applications, Extensions and Open Questions

Cost-sharing games have been studied under the following (underlying) assumption: given *any* subset Q of users, it is possible to provide the service to *exactly* those users in Q.

This hypothesis cannot be taken for granted in several applications. Indeed, consider the following (simple) scenarios:

Fig. 1(a). A network connecting a source node s to another node t, and $n \geq 2$ users all sitting on node t. If the source s transmits to any of them, then all the others will also receive it. (Consider the scenario in which there is no encryption and one users can "sniff" what is sent to the others. This problem is a variant of the games considered in [8,4,3,5].

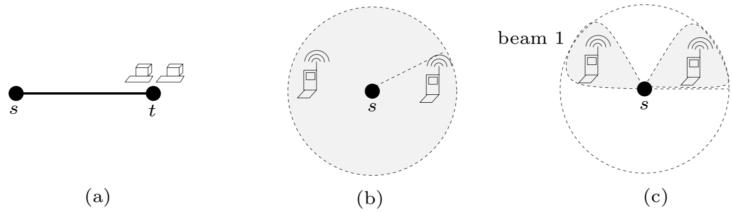

Fig. 1. (a) A variant of the Steiner tree game in [3]; (b) A variant of the wireless multicast game in [1]; (c) Another variant of the wireless multicast game obtained by considering stations with switched-beams antennae and limited battery capacity

Fig. 1(b). The wireless multicast game in which a source station s and $n \geq 2$ other stations/users located all in the transmission range of s. Similarly to the previous example, station s can only choose to transmit to *all* of the them or to none. This game is a variant of the one in [1], where the authors implicitly assume that stations/users receiving a physical signal are not able to get the transmission.

Fig. 1(c). As above, but now the source s uses a switched beam antenna: the coverage area is divided into independent sectors or *beams*. The energy spent by s depends on the number of used sectors. It may be the case that the battery level of s is sufficient to reach one user, but not both.

The first two problems are equivalent to a simple generalized cost-sharing game with $\mathcal{P} = \{U, \emptyset\}$. The latter, instead, corresponds to the case $U \notin \mathcal{P}$. Corollary 1 implies that none of the three instances above admits an α-BB, NPT, VP, CS, and GSP mechanism. The same holds for several natural variants of cost-sharing games studied in the literature [10,9,3,1,12,6,2,5], where connectivity games on graphs allow more than one user per node but no "encryption": either all users in that node are serviced or none.

A similar negative result holds if the service provider is not able to service all of its potential customers (i.e., $U \notin \mathcal{P}^{\mathsf{A}}$), as in the third example. This requirement implies some lower bounds on *polynomial-time* mechanisms which

Table 1. A summary of upper/lower bounds on 'α' for mechanisms satisfying α-BB, NPT, VP, CS and GSP. Quantity $\rho(\mathcal{X})$ is the best approximation guarantee of any polynomial-time algorithm approximating $C(\cdot)$ over $\mathcal{X} \subseteq 2^U$). Results marked '*' holds in general (i.e., for non-upper continuous mechanisms too).

(Generalized) Cost-Sharing Games		**Upper Continuous Mechanisms**	
		any (non polytime)	poly-time
With Sequential Algorithms	$\mathcal{P} = 2^U$	1 [14,2]	$\alpha \leq \rho(2^U)$ [14]
			$\alpha \geq \rho(\{U\})$ [Cor. 2]*
	$\mathcal{P}^\sigma \subseteq \mathcal{P}$	1 [14,2]	$\alpha \leq \rho(\mathcal{P}^\sigma)$ [14]
			$\alpha \geq \rho(\{U\})$ [Cor. 2]*
With No Sequential Algorithm	$\mathcal{P}^\sigma \not\subseteq \mathcal{P}$	unbounded [Cor. 4]	unbounded [Cor. 4]

relate the computational hardness of approximating $C(U)$ to the factor α-BB condition (Corollary 2).

If one ignores computational issues, than Corollary 5 states that, for upper continuous mechanisms, generalized games which are "solvable" are all and only those that admit a sequential algorithm. Here the factor α plays no role. In other words, if we stick to properties NPT, VP, CS and GSP only, then it makes sense to relax BB to α-BB only for computational reasons. This contrasts with prior results in [2] where adding a "fairness" requirement (i.e., no free riders) then paying a factor $\alpha > 1$ is necessary (and sometimes sufficient) for upper continuous mechanisms, regardless of their running time.

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