Succinct Representations of Model Based Belief Revision*

(Extended Abstract)

Paolo Penna

Dipartimento di Matematica, Università di Roma "Tor Vergata",
penna@mat.uniroma2.it

Abstract. In this paper, following the approach of Gocic, Kautz, Papadimitriou and Selman (1995), we consider the ability of belief revision operators to succinctly represent a certain set of models. In particular, we show that some of these operators are more efficient than others, even though they have the sane model checking complexity. We show that these operators are partially ordered, i.e. some of them are not comparable. We also strengthen some of the results by Cadoli, Donini, Liberatore and Shaerf (1995) by showing that for some of the so called "model based" operators, a polynomial size representation does not exist even if we allow the new knowledge base to have a non polynomial time model checking (namely, either in NP or in co-NP). Finally, we show that Dalal's and Weber's operators can be compiled one into the other via a formalism whose model checking is in NP. All of our results also hold when iterated revision, for one or more of the operators, is considered.

1 Introduction

Several formalisms for knowledge representation and nonmonotonic reasoning have been proposed and studied in the literature. Such formalisms often give rise to intractable problems, even when propositional versions of such formalisms are considered (see [7] for a survey).

Knowledge compilation aims to avoid these difficulties through an off-line process where a given knowledge base is compiled into an equivalent one that supports queries more efficiently. The feasibility of the above approach has been deeply investigated depending on several factors such as: the formalism used for the original and resulting knowledge base, the kind of equivalence we require, and so on (see [4] for a survey). For example, let us consider the propositional version of *circumscription* ($\mathcal{CIRC}$), a well known form of nonmonotonic reasoning introduced in the AI literature in [16, 17]. Informally, $\mathcal{CIRC}(T)$ denotes those truth assignments that satisfy T and that have a "minimal" set of variables mapped into 1. The idea behind minimality is to assume that a fact is false whenever possible. In particular, we represent a truth assignment as a subset m of variables of

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T (those mapped into 1) and we say that m is a *model* if the corresponding truth assignment satisfies T. Then, $\mathcal{CIRC}(T)$ contains only the models of T that are minimal w.r.t. set inclusion (see Sect. 1.2 for a formal definition). Although it is possible to explicitly represent all the models in $\mathcal{CIRC}(T)$, this representation in general has size exponential in the size of T. So, a shorter (implicit) representation is given by the propositional formula T. However, representing the set of models $\mathcal{CIRC}(T)$ simply as T yields an overhead from the computational point of view. For instance, given T and a subset m of its variables, deciding whether $m \in \mathcal{CIRC}(T)$ (*model checking*) is an co-NP-complete problem [2]. Notice that in the classical propositional logic ($\mathcal{PL}$) a formula F simply represents all of its models, so model checking for $\mathcal{PL}$ is clearly in P. Similarly, deciding whether a formula logically follows from $\mathcal{CIRC}(T)$ (*inference*) is a $\prod_2^p$-complete problem [9], while inference for $\mathcal{PL}$ is co-NP-complete. A natural question is therefore: is it possible to "translate" $\mathcal{CIRC}(T)$ into a *propositional formula* F and then use F (instead of T) to solve the model checking problem in time polynomial in $|T|$? Clearly such translation cannot be performed in polynomial time unless P = NP (that is why we need to do it off-line). Additionally, a necessary condition is the size of F to be polynomially bounded in the size of T. A negative answer to this question has been given in [6] where the authors proved that, in general, $|F|$ is not polynomially bounded in $|T|$. Informally, this is due to the fact that $\mathcal{CIRC}$ allows for representations of the information (i.e. a set of models) that are much more "succinct" than any equivalent representation in $\mathcal{PL}$ (see [6] for more formal definitions of what 'equivalent' means).

The above idea of compiling one formalism into another has been extended in [13] where the relative *succinctness* – also known as *compactness* or *space efficiency* – of several propositional logical formalisms has been investigated. The way two formalisms can be compared is the following. A formalism $\mathcal{F}_1$ is more efficient than a formalism $\mathcal{F}_2$ if: (a) $\mathcal{F}_1$ can be compiled into $\mathcal{F}_2$ and (b) $\mathcal{F}_2$ cannot be compiled into $\mathcal{F}_1$, where the compilation requires the new knowledge base being model equivalent and having size polynomial w.r.t. the original one. It is worth observing that, by one hand, *succincteness implies non-compactability.* By the other hand, the converse does not always hold since it might be the case that $\mathcal{F}_1$ and $\mathcal{F}_2$ cannot be compiled one into the other, i.e. they are *not comparable.* A somehow surprising result of [13] is that formalisms having the same model checking time complexity are instead totally ordered in terms of succinctness. In this case, succinctness becomes crucial in choosing one formalism instead of another to represent the knowledge.

Another important aspect of nonmonotonic reasoning is that we have to deal with uncertain and/or incomplete information. Several criteria for updating and/or revising a knowledge base have been proposed [1, 11, 12, 18, 21, 22, 24]. Suppose we have a knowledge base T and a new piece of information, represented by a formula P, is given. It might be the case that T and P are not consistent. In this case the *revision* of T with P, denoted as $T \circ P$, contains those models of P defined by means of a belief revision operator '$\circ$'. The so called *model based* operators define the set of models of $T \circ P$ as those models of

P that are "close" to the models of T. To different definitions of closeness correspond different revision operators. *Syntax based* approaches are instead defined in terms of syntactic operations on the knowledge base T. In general, model based approaches are preferred to syntax based ones because of their *syntax irrelevance*, i.e. revising two logical equivalent knowledge bases T and T' with a formula P always yields the same set of models. Also in this case model checking and inference become harder than in $\mathcal{PL}$ [15, 8] (see also Table 1). This is a first motivation for investigating compilability of belief revision into $\mathcal{PL}$. Moreover, since we are dealing with revision of knowledge, it is often required to explicitly compute a propositional formula T' equivalent to $T \circ P$, that is the revised knowledge base. In such a case it might be desirable not to have an exponential increase in the size of the original knowledge base. Unfortunately, for several revision operators non-compactability results have been proved in [5]. In the same paper also a weaker kind of equivalence has been considered: *query equivalence*. In this case the compilation does not preserve the set of models but just the set of formulas that logically follow. So, it can be used for inference but not for model checking. In Table 1 we summarize both the complexity and the compactability results proved for several belief revision operators, both model and syntax based (Ginsberg's and WIDTIO).

It is interesting to observe that some revision operators and $\mathcal{CIRC}$ have similar properties. For instance, Ginsberg's operator and $\mathcal{CIRC}$ have the same time complexity and the same compactability properties (see [2, 9, 6] for the results on $\mathcal{CIRC}$). It is therefore natural to ask whether this is a chance or not. A first study of relationships between belief revision and $\mathcal{CIRC}$ has been done in [23] where the author remarked similarities between $\mathcal{CIRC}$ and her operator. Subsequently, in [14] the authors pointed out interesting connections between $\mathcal{CIRC}$ and several belief revision operators, thus extending the result of [23]. In particular, they proved that $\mathcal{CIRC}$ can be compactly represented by means of several belief revision operators, i.e. given a propositional formula F, two formulas T and P, of size polynomial w.r.t. $|F|$, exist such that $T \circ P$ is logically equivalent to $\mathcal{CIRC}(F)$. As remarked in [14], this allows to import results from one field into the other. For example, the above mentioned result combined with the non-compactability results of $\mathcal{CIRC}$ can be used to prove several of the negative results in [5]. Also inverse reductions have been investigated, i.e. compiling belief revision into $\mathcal{CIRC}$, but in this case query equivalence (instead of model equivalence) is considered. In [3], among other results, a precise characterizations of compactability properties of Ginsberg's operators is given. In fact, it can be compiled in $\mathcal{CIRC}$ and vice versa. Additionally, such result also holds for the case of iterated revision, i.e. when a polynomial number of revision steps is considered, by making use of the fact that also in this case the model checking is in co-NP [10].

Finally, we remark that all of the non compilability results in [5, 6, 3] and some of those in [13] relies on the standard hypothesis that the polynomial hierarchy does not collapse. Moreover, the results in [6] hold if and only if this hypothesis is true.

Operator	**Complexity**		**Compactability**	
	Model Checking [15]	**Inference** [8]	**Model** [5]	**Query** [5]
Ginsberg	co-NP-complete	$\prod_2^p$-complete	No	No
Winslett, Borgida, Forbus, Satoh.	$\sum_2^p$-complete	$\prod_2^p$-complete	No	No
Dalal	$\mathrm{P}^{\mathrm{NP}[O(\log n)]}$-comp.	$\mathrm{P}^{\mathrm{NP}[O(\log n)]}$-comp.	No	Yes
Weber	$\sum_2^p$-complete	$\prod_2^p$-comp. [19] & [8]	No	Yes
WIDTIO	$\sum_2^p$-complete	$\prod_2^p$-comp. [19] & [8]	Yes	Yes

Table 1. Previous results: complexity and compactability of belief revision operators.

1.1 Results of the Paper

In this paper we give a better characterization of (non) compactability properties of belief revision and we provide important connections between such operators and $\mathcal{CIRC}$. We consider the model based revision operators in Table 1 and we compare their space efficiency with that of $\mathcal{CIRC}$, as well as their relative compactness. In particular, we show that, for some model based operators (Winslett's, Borgida's, Forbus's, and Satoh's) belief revision is more difficult to be compiled than $\mathcal{CIRC}$, Ginsberg's, Dalal's or Weber's revision. For the latter two operators we give a precise characterization of their compactability properties. Our results significantly strengthen several non-compactability results in [5] and the results of [15]. Moreover, they provide an intuitive explanation of the (non) compactability results when query equivalence is considered. The results are obtained under the assumption that the polynomial hierarchy does not collapse.

To this aim, we introduce a formalism, denoted as $\overline{\mathcal{CIRC}}$, whose model checking is in NP and is not comparable to $\mathcal{CIRC}$. Roughly speaking, $\overline{\mathcal{CIRC}}$ can be seen as the "complement" of $\mathcal{CIRC}$, i.e. $\overline{\mathcal{CIRC}}$ corresponds to the set of *non minimal models* of a propositional formula. In Fig. 1 we show relationships among revision operators and their space efficiency with respect to $\mathcal{PL}$, $\mathcal{CIRC}$ and $\overline{\mathcal{CIRC}}$, where one way arrows represent the fact that one operator is strictly more succinct then another. The results are consequences of previously known results combined with the following two:

– $\overline{\mathcal{CIRC}}$ can be compiled into model based operators;
– Dalal's and Weber's operators can be compiled into $\overline{\mathcal{CIRC}}$.

As a consequence we have that Winslett's, Borgida's, Forbus's and Satoh's operators are more succinct than all the other operators, and Ginsberg's operator is not comparable to Dalal's or Weber's one. Moreover, Dalal's and Weber's can be reduced each other via $\overline{\mathcal{CIRC}}$. This yields a *precise characterization* of their space efficiency w.r.t. $\mathcal{CIRC}$ and the other belief revision operators. Additionally, the fact that Dalal's and Weber's operators are equivalent to $\overline{\mathcal{CIRC}}$ gives an intuitive explanation of their query compactability properties [5] (see Table 2).

Motivated by the non-compactability results of [5], we attempt to find a trade-off between compactability and the complexity of model checking of the

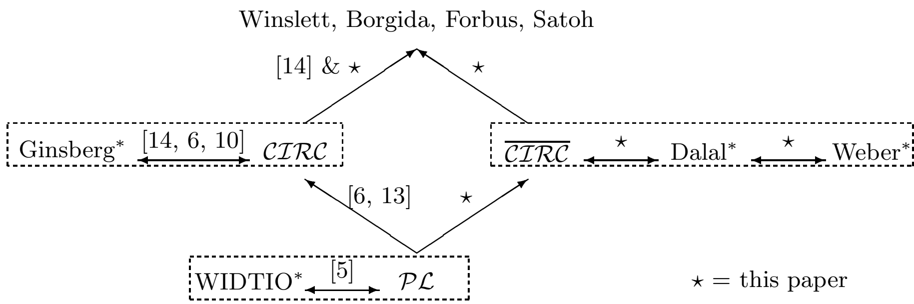

Fig. 1. Results of the paper: the relative succinctness of belief revision operators, where '$*$' means that the result also holds for iterated revision.

knowledge base in which the original one is compiled. In particular, we consider the following question:

> Can we succinctly represent a revised knowledge base by means of $\mathcal{CIRC}$? More generally, can it be compiled into a knowledge base whose model checking is either in NP or in co-NP?

Since model checking for model based operators is harder than any problem in NP or in co-NP, a positive answer to the above question can be used to make model checking easier through an off-line preprocessing of compilation. In Table 2 we summarized the obtained results, which follow from properties of $\mathcal{CIRC}$ and $\overline{\mathcal{CIRC}}$.

Operator	**Compactable into a knowledge base whose model checking is in**	
	NP	co-NP
Ginsberg	No Corollary 3 & [14]	Yes, also iterated [15, 6, 10]
Winslett	No Corollary 3 & [14]	No Corollary 2
Borgida	No Corollary 3 & [14]	No Corollary 2
Satoh	No Corollary 3 & [14]	No Corollary 2
Dalal	Yes, also iterated [5], also Theorem 4	No Corollary 2
Weber	Yes, also iterated [5], also Theorem 4	No Corollary 2

Table 2. Results of the paper: compactability w.r.t. model checking time complexity of the new knowledge base.

It is worth observing that:

- None of the model based operators admits compact representations whose model checking is in co-NP.
- Dalal's and Weber's operators admit compact representations whose model checking is in NP, even when iterated revision is considered.
- None of the other model based operators admits compact representations whose model checking is in NP.

The latter result strengthen the negative result proved in [5] in that no model equivalent knowledge base exists even when we allow its model checking to be either in NP or in co-NP. Additionally, it is not possible to compile a model based revision operator into $\mathcal{CIRC}$, thus implying that the result of [15] (which holds in the case of query equivalence) cannot be extended to model equivalence.

We emphasize that the compactness of belief revision operators, in general, does not seem to depend on either the complexity of inference and model checking or the previously known compactability results. For instance, Winslett's and Weber's ones have the same complexity (see Table 1) while they are ordered in terms of space efficiency (see Fig. 1). Additionally, Dalal's and Weber's, that have different complexity, can be compiled one into the other, instead. The reducibility of those two operators to $\overline{\mathcal{CIRC}}$ also gives an intuitive explanation of their compactability properties (see Table 1) which, actually, are the same as $\overline{\mathcal{CIRC}}$

Due to lack of space some of the proofs of the above results will be omitted or only sketched in this extended abstract.

1.2 Preliminaries

Given a propositional formula F and given a truth assignment m to the variables of F, we say that m is a *model* of F if m satisfies F. Models will be denoted as sets of variables (those mapped into 1). We denote by $\mathcal{M}(F)$ the set of models of F. A *theory* is a set T of propositional formulas. The set of models $\mathcal{M}(T)$ of the theory T is the set of models that satisfy all of the formulas in T. If $\mathcal{M}(F) \neq \emptyset$ then the formula is satisfiable. Similarly, a theory T is consistent if $\mathcal{M}(T) \neq \emptyset$. We use $a \rightarrow b$ and $a \leftrightarrow b$ as a shorthand for $\neg a \vee b$ and $(a \wedge b) \vee (\neg a \wedge \neg b)$, respectively. Given two models m and n, we denote by $m \Delta n$ their symmetric difference. Given a set of sets $\mathcal{S}$, we denote by $\min_{\subseteq} \mathcal{S}$ (respectively, $\max_{\subseteq} \mathcal{S}$), the minimal (respectively, maximal) subset of $\mathcal{S}$ w.r.t. set inclusion. The *circumscription* of a propositional formula F (denoted by $\mathcal{CIRC}(F)$) is defined as

$$\mathcal{CIRC}(F) \doteq \{m \in \mathcal{M}(F) \mid \forall m' \subset m, m' \notin \mathcal{M}(F)\} = \min_{\subseteq} \mathcal{M}(F).$$

Given a theory T and a propositional formula P, we denote by $T \circ P$ the theory T revised with P according to some belief revision operator $\circ$. We distinguish the following belief revision operators:

Ginsberg. Let $\mathcal{W}(T,P) \doteq \max_{\subseteq} \{T' \subseteq T \mid T' \cup \{P\} \not\models \bot\}$. Then $T \circ_G P \doteq \{T' \cup \{P\} \mid T' \in \mathcal{W}(T,P)\}$.

Winslett. Let $\mu(m,P) \doteq \min_{\subseteq} \{m\Delta n \mid n \in \mathcal{M}(P)\}$. Then,
$\mathcal{M}(T \circ_{Win} P) \doteq \{n \in \mathcal{M}(P) \mid \exists m \in \mathcal{M}(T) : m\Delta n \in \mu(m,P)\}$.

Borgida. It is defined as $\circ_{Win}$ if $T \cup \{P\}$ is not consistent, and it is defined as $T \cup \{P\}$ otherwise.

Forbus. For any two models m_1 and m_2 let $d(m_1,m_2) = |m_1 \Delta m_2|$. Also let $k_{m,P} = \min\{d(m,n) : n \in \mathcal{M}(P)\}$. Then,
$\mathcal{M}(T \circ_F P) \doteq \{n \in \mathcal{M}(P) \mid \exists m \in \mathcal{M}(T) : d(m,n) = k_{m,P}\}$.

Satoh. Let $\delta(T,P) \doteq \min_{\subseteq}\{\bigcup_{m \in M(P)} \mu(m,P)\}$. Then,
$\mathcal{M}(T \circ_S P) \doteq \{n \in \mathcal{M}(P) \mid \exists m \in \mathcal{M}(T) : m\Delta n \in \delta(T,P)\}$.

Dalal. Let $k_{T,P} = \min\{k_{m,P} \mid m \in \mathcal{M}(T)\}$. Then,
$\mathcal{M}(T \circ_D P) \doteq \{n \in \mathcal{M}(P) \mid \exists m \in \mathcal{M}(T) : d(m,n) = k_{T,P}\}$.

Weber. Let $\Omega = \bigcup \delta(T,P)$, i.e. Ω contains all of the variables appearing on a minimal difference between models of T and model of P. Then,
$\mathcal{M}(T \circ_{Web} P) \doteq \{n \in \mathcal{M}(P) \mid \exists m \in \mathcal{M}(T) : m\Delta n \subseteq \Omega\}$.

An *advise taking Turing machine* is a Turing machine that can access an advice $a(n)$, i.e. an "oracle" whose output depends *only* on the size n of the input. The class NP/poly is the class of those languages that are accepted by a nondeterministic Turing machine with an advice of polynomial size (see [20] for a formal definition). The class co-NP/poly is similarly defined. In [25] non-uniform classes such as NP/poly and the polynomial hierarchy have been related. In particular, it has been proved that if NP $\subseteq$ co-NP/poly or co-NP $\subseteq$ NP/poly then the polynomial hierarchy (denoted by PH) collapses at the third level, i.e. PH $= \sum_3^p$ (see [20] for a formal definition of those concepts), which is considered *very unlikely* in the complexity community.

2 The Complemented Circumscription and Its Properties

In this section we introduce $\overline{\mathcal{CIRC}}$ and state its basic (non) compactability properties that will be used in the rest of the paper. To this aim we first introduce a model equivalence preserving reduction used in [3] and we assume that a knowledge base K in a formalism $\mathcal{F}$ represents a set of models $\mathcal{F}(K)$.

Definition 1 ([3]). *Given two logical formalisms $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_1 \mapsto \mathcal{F}_2$ if the following holds: for each knowledge base K_1 in $\mathcal{F}_1$, there exists a knowledge base K_2 in $\mathcal{F}_2$ and a polynomial time computable function g_{K_1} such that (i) for any set of variables m_1, $m_1 \in \mathcal{F}_1(K_1) \Leftrightarrow g_{K_1}(m_1) \in \mathcal{F}_2(K_2)$; (ii) $|K_2|$ is polynomially bounded in $|K_1|$.*

The above definition implies that, once we have computed (off-line) the formula K_2, we can decide whether m_1 is a model of K_1, by checking if $g_{K_1}(m_1)$ is a model of K_2. Additionally, $g_{K_1}(m_1)$ can be computed in polynomial time. Finally, the '$\mapsto$' relation is transitive.

Definition 2. *We denote by $\overline{\mathcal{CIRC}}(F)$ the set of non minimal models of a propositional formula F, that is*

$$\overline{\mathcal{CIRC}}(F) = \mathcal{M}(F) \setminus \mathcal{CIRC}(F) = \{m \in \mathcal{M}(F) \mid \exists m' \in \mathcal{M}(F) : m' \subset m\}.$$

Lemma 1 ([6, 13]). *For any n a propositional formula F_n (of size polynomial in n) exists such that for every n-variables 3CNF propositional formula f there exists a model m_f (computable in polynomial time) such that*

$$f \text{ is unsatisfiable} \Leftrightarrow m_f \in \mathcal{CIRC}(F_n).$$

The above lemma states that the circumscription of a formula F_n is able to capture all of the unsatisfiable 3CNF formulas with n variables (notice that F_n depends only on n). As a consequence we obtain the following result, whose proof is similar to non compilability proofs given in [6, 13].

Theorem 1. *The following hold: (i) $\mathcal{CIRC} \mapsto \overline{\mathcal{CIRC}} \Rightarrow$ co-NP $\subseteq$ NP/poly; (ii) $\overline{\mathcal{CIRC}} \mapsto \mathcal{CIRC} \Rightarrow$ NP $\subseteq$ co-NP/poly.*

The above theorem can be easily generalized to any two formalisms whose model checking is in co-NP and NP, respectively. Thus, the following corollary holds.

Corollary 1. *Let $\mathcal{F}_{\text{co-NP}}$ and $\mathcal{F}_{\text{NP}}$ be any two formalism whose model checking is in* co-NP *and* NP*, respectively. Unless the polynomial hierarchy collapses at the third level, the following two hold: (i) $\mathcal{CIRC} \not\mapsto \mathcal{F}_{\text{NP}}$; (ii) $\overline{\mathcal{CIRC}} \not\mapsto \mathcal{F}_{\text{co-NP}}$.*

In the rest of the paper we will make use of the above result and thus we will always assume PH $\neq \sum_3^p$.

3 Reducing $\overline{\mathcal{CIRC}}$ to Belief Revision

In this section we provide some reductions from $\overline{\mathcal{CIRC}}$ to any of the model based belief revision operators. As a consequence we have that none of such operators can be compactly represented by $\mathcal{CIRC}$ or by $\circ_G$.

Theorem 2. *$\overline{\mathcal{CIRC}} \mapsto \circ_{Win}$, $\overline{\mathcal{CIRC}} \mapsto \circ_B$, $\overline{\mathcal{CIRC}} \mapsto \circ_F$.*

Proof. We will prove the theorem only for the $\circ_{Win}$ operator, since the proof can be easily adapted to the other two operators. Let F be a propositional formula over the variables $x_1, \ldots, x_n$. We show that two formulas T and P of polynomial size exist such that $\mathcal{M}(T \circ_{Win} P) = \overline{\mathcal{CIRC}}(F)$.

Let $y_1, \ldots, y_n$ be a set of new variables in correspondence one-to-one with $x_1, \ldots, x_n$ and let m^y be the set $\{y_i | x_i \in m\}$. We construct two formulas T and P over the set of variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that

$$\mathcal{M}(T) = \{m_1 \cup m_2^y \;\mid\; m_1, m_2 \in \mathcal{M}(F), m_2 \subset m_1\}$$

and $\mathcal{M}(P) = \mathcal{M}(F)$. Let us observe that if no two models $m_1, m_2 \in \mathcal{M}(F)$ exist such that $m_2 \subset m_1$, then T is not satisfiable. We will see in the sequel how to deal with that case. Thus, let us suppose $\mathcal{M}(T) \neq \emptyset$ and let

$$T = F \wedge F[x_i/y_i] \wedge \underbrace{\neg\left(\bigwedge_{i=1}^{n} x_i \rightarrow y_i\right)}_{m_1 \not\subseteq m_2} \wedge \underbrace{\left(\bigwedge_{i=1}^{n} y_i \rightarrow x_i\right)}_{m_2 \subseteq m_1}$$

and $P = F \wedge \bigwedge_{i=1}^{n} \neg y_i$. We now prove that $m \in \mathcal{M}(T \circ_{Win} P) \Leftrightarrow m \in \overline{\mathcal{CIRC}}(F)$.

($\Rightarrow$) By the definition of $\circ_{Win}$ we have that a model $m^T \in \mathcal{M}(T)$ exists such that $m \in \mu(m^T, P)$. Let $m^T = m_1 \cup m_2^y$, where $m_1, m_2 \in \mathcal{M}(F)$ and $m_2 \subset m_1$. Let us first observe that, since m does not contain any variable y_i,

$$m \Delta m^T = (m \Delta m_1) \cup m_2^y.$$

We now prove that $m = m_1$. Suppose, by contradiction, that $m \Delta m_1 \neq \emptyset$. Then, $m_1 \Delta m^T = m_2^y \subset m \Delta m^T$, which implies that $m \notin \mu(m^T, P)$, thus a contradiction. So, $m_2 \subset m_1 = m$, that is $m \in \overline{\mathcal{CIRC}}(F)$.

($\Leftarrow$) There exists $m_2 \in \mathcal{M}(F)$ such that $m_2 \subset m$. Let $m^T = m \cup m_2^y$. Clearly $m^T \in \mathcal{M}(T)$. Suppose by contradiction that $m \notin \mu(m^T, P)$. Thus, an $m_1 \in \mathcal{M}(P)$ exists such that $m_1 \Delta m^T = (m_1 \Delta m) \cup m_2^y \subset m \Delta m^T = m_2^y$, thus a contradiction.

We now consider the case in which no two models $m_1, m_2 \in \mathcal{M}(F)$ exist such that $m_2 \subset m_1$. To this aim, we have to slightly modify the above construction and consider the formula $F' = F \vee \bigwedge_{i=1}^{n+1} x_i$, where x_{n+1} is a new variable. Let T' and P' be the formulas obtained by replacing F with F' in the definition of T and P, respectively. Let $\overline{x} = \{x_1, \ldots, x_{n+1}\}$. We then have that

$$\mathcal{M}(T' \circ_{Win} P') = \overline{\mathcal{CIRC}}(F') = \overline{\mathcal{CIRC}}(F) \cup \{\overline{x}\}.$$

Finally, the above reduction also apply to $\circ_F$, while it can be easily adapted for $\circ_B$ (it suffices to guarantee that $T' \wedge P'$ is not consistent). Hence, the theorem follows.

Theorem 3. *$\overline{\mathcal{CIRC}} \mapsto \circ_D$, $\overline{\mathcal{CIRC}} \mapsto \circ_{Web}$, $\overline{\mathcal{CIRC}} \mapsto \circ_S$.*

Proof. (*sketch of*) First of all we slightly modify the formulas T and P of Theorem 2 as follows:

$$T = F' \wedge F'[x_i/y_i] \wedge \neg \underbrace{\left(\bigwedge_{i=1}^{n+1} x_i \rightarrow y_i \right)}_{m_1 \not\subseteq m_2} \wedge \underbrace{\left(\bigwedge_{i=1}^{n+1} y_i \rightarrow x_i \right)}_{m_2 \subseteq m_1} \wedge \left(\bigwedge_{i=1}^{n+1} \neg y_i \leftrightarrow z_i \right),$$

and $P = F' \wedge \bigwedge_{i=1}^{n+1} \neg y_i \bigwedge_{i=1}^{n+1} \neg z_i$, where F' is defined as in the proof of Theorem 2. In the case of $\circ_D$, the proof is a consequence of the following claims:

Claim 1: $k_{T,P} = n + 1$.
Claim 2: *For all $m \in \overline{\mathcal{CIRC}}(F)$, there exists $m^T \in \mathcal{M}(T)$ such that $d(m, m^T) = n + 1$.*
Claim 3: *For any $m \in \mathcal{CIRC}(F)$ and for all $m^T \in \mathcal{M}(T)$, $d(m, m^T) > n + 1$.*

As far as $\circ_{Web}$ and $\circ_S$ concerns, we first observe that $\delta(T, P)$ does not contain any variable x_i. Moreover, it is easy to see that, for any $n \in \mathcal{CIRC}(F)$, and for any $m^T \in \mathcal{M}(T)$, $n \Delta m^T$ contains at least one variable x_i. This proves the theorem.

4 (Non) Compactability of Model Based Revision

In this section we consider the problem of compiling the revised knowledge base into a model equivalent one that has model checking either in NP or co-NP. To this aim we will denote by $\mathcal{F}_{\mathrm{NP}}$ and $\mathcal{F}_{\text{co-NP}}$ any two formalisms[1] whose model checking is in NP and co-NP, respectively.

Let us first observe that an immediate consequence of the reductions given in Sect. 3 and of Corollary 1 is the following fact.

Corollary 2. *For any* $\circ \in \{\circ_{Win}, \circ_B, \circ_F, \circ_S, \circ_D, \circ_{Web}\}$, $\circ \not\mapsto \mathcal{F}_{\text{co-NP}}$.

The above result implies that such operators cannot be represented by means of $\mathcal{CIRC}$. Motivated by this fact we ask whether it is possible to obtain compact representations of $\mathcal{M}(T \circ P)$ by means of F_{NP}. In this case, we show that the situation is more tangled.

We first consider Dalal's and Weber's revision and show that they admit a compact representation by means of $\overline{\mathcal{CIRC}}$.

The main idea of the reductions is that both $k_{T,P}$ and Ω can be represented in polynomial space ($k_{T,P}$ is an integer and $|\Omega| \leq n$). Moreover, once those two entities have been computed (off-line), then the problem of deciding $m \in \mathcal{M}(T \circ P)$ is in NP for both the operators.

Theorem 4. $\circ_D \mapsto \overline{\mathcal{CIRC}}$, $\circ_{Web} \mapsto \overline{\mathcal{CIRC}}$.

The above result can be easily extended to the case of a polynomial number of revision steps. Notice that a different proof can be derived by making use of the fact that such two operators are query compactable [5].

We now consider the other revision operators. To this aim we combine the results proved in Sect. 3 with the results given in [14]. In particular we exploit the fact that such operators can be used to represent $\mathcal{CIRC}$.

Theorem 5 ([14]). *For any* $\circ \in \{\circ_{Win}, \circ_B, \circ_F, \circ_S\}$, $\mathcal{CIRC} \mapsto \circ$

The above theorem combined with Corollary 1 yields the following result.

Corollary 3. *For any* $\circ \in \{\circ_{Win}, \circ_B, \circ_F, \circ_S\}$, $\circ \not\mapsto \mathcal{F}_{\mathrm{NP}}$.

5 Succinctness of Belief Revision

We compare the space efficiency of the belief revision operators and consider the problem of compiling one operator into another. By combining our results with previously known results we will obtain the partial ordering shown in Fig. 1.

To this aim we first introduce the following notation.

[1] In this case the term formalism is quite general, since it refers to any representation of a set of models.

Definition 3. *For any two logical formalisms* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*: (i)* $\mathcal{F}_1 \prec \mathcal{F}_2$ *if both* $\mathcal{F}_1 \mapsto \mathcal{F}_2$ *and* $\mathcal{F}_2 \not\mapsto \mathcal{F}_1$*; (ii)* $\mathcal{F}_1 \approx \mathcal{F}_2$ *if* $\mathcal{F}_1 \mapsto \mathcal{F}_2$ *and* $\mathcal{F}_2 \mapsto \mathcal{F}_1$*; (iii)* $\mathcal{F}_1 \not\approx \mathcal{F}_2$ *if both* $\mathcal{F}_1 \not\mapsto \mathcal{F}_2$ *and* $\mathcal{F}_2 \not\mapsto \mathcal{F}_1$*.*

All of the following results are easy consequences of Theorem 3, Theorem 4 and Corollary 3. We first compare $\circ_D$ and $\circ_{Web}$ operators with the other model based ones. All of the results also hold for a polynomial number of revision steps of $\circ_D$ or $\circ_{Web}$.

Corollary 4. *For any* $\circ \in \{\circ_{Win}, \circ_B, \circ_F, \circ_S\}$, $\circ_D \approx \circ_{Web} \approx \overline{\mathcal{CIRC}} \prec \circ$.

Corollary 5. *For any* $\circ \in \{\circ_D, \circ_{Web}\}$, $\mathcal{CIRC} \not\approx \circ$ *and* $\circ_G \not\approx \circ$.

6 Conclusions and Open Problems

We have shown that belief revision operators with the same model checking and inference complexity have different behaviours in terms of compilability and space efficiency. We precisely characterized the space efficiency of $\circ_D$ and $\circ_{Web}$ which, following the definitions of [3], are model-NP-complete. Moreover, our results combined with those in [14] imply that $\circ_{Win}$, $\circ_B$, $\circ_F$ and $\circ_S$ are both model-NP-hard and model-co-NP-hard.

The first problem left open is that of finding similar characterizations for the latter operators, as well as that of understanding their relative space efficiency. More generally, it could be interesting to investigate relationships with other formalisms considered in [13, 3] such as default logic, model preference and autoepistemic logic. Furthermore, compactability results for the case of iterated revision are not known. Do these operators became even harder to be compacted when more than one step of revision is considered?

It is interesting to observe that a different situation occurs when query equivalence is considered. Indeed, in [14] the authors proved that in this case $\circ_G$ and $\circ_S$ can be reduced one to the other and $\circ_{Win}$ can be reduced to both.

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