The Power Range Assignment Problem in Radio Networks on the Plane
(Extended Abstract)

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Abstract. Given a finite set S of points (i.e. the stations of a radio network) on the plane and a positive integer $1 \leq h \leq |S| - 1$, the 2D Min h R. Assign. problem consists of assigning transmission ranges to the stations so as to minimize the total power consumption provided that the transmission ranges of the stations ensure the communication between any pair of stations in at most h hops.
We provide a lower bound on the total power consumption $\mathsf{opt}_h(S)$ yielded by an optimal range assignment for *any* instance (S, h) of 2D Min h R. Assign., for any positive constant $h > 0$. The lower bound is a function of $|S|$, h and the minimum distance over all the pairs of stations in S. Then, we derive a constructive upper bound for the same problem as a function of $|S|$, h and the maximum distance over all the pairs of stations in S (i.e. *the diameter* of S). Finally, by combining the above bounds, we obtain a polynomial-time approximation algorithm for 2D Min h R. Assign. restricted to *well-spread* instances, for any positive constant h.
Previous results for this problem were known only in special 1-dimensional configurations (i.e. when points are arranged on a line).

Keywords: Approximation Algorithms, Lower Bounds, Multi-Hop Packet Radio Networks, Power Consumption.

1 Introduction

A *Multi-Hop Packet Radio Network* [6] is a finite set of radio stations located on a geographical region that are able to communicate by transmitting and receiving radio signals. A transmission range is assigned to each station s and any other station t within this range can directly (i.e. by one *hop*) receive messages from s. Communication between two stations that are not within their respective ranges can be achieved by *multi-hop* transmissions. In general, Multi-Hop Packet Radio Networks are adopted whenever the construction of more traditional networks is impossible or, simply, too expensive.

H. Reichel and S. Tison (Eds.): STACS 2000, LNCS 1770, pp. 651–660, 2000.

It is reasonably assumed [6] that the power P_t required by a station t to correctly transmit data to another station s must satisfy the inequality

$$\frac{P_t}{d(t,s)^\beta} > \gamma \tag{1}$$

where $d(t,s)$ is the distance between t and s, $\beta \geq 1$ is the *distance-power gradient*, and $\gamma \geq 1$ is the *transmission-quality* parameter. In an ideal environment (see [6]) $\beta = 2$ but it may vary from 1 to more than 6 depending on the environment conditions of the place the network is located. In the rest of the paper, we fix $\beta = 2$ and $\gamma = 1$, however, our results can be easily extended to any $\beta, \gamma \geq 1$.

Given a set $S = \{s_1, \ldots, s_n\}$ of radio stations on an Euclidean space, a *range assignment* for S is a function $r : S \to \mathcal{R}^+$, and the *cost* of r is defined as

$$\mathsf{cost}(r) \;=\; \sum_{i=1}^{n} r(s_i)^2.$$

As defined in the abstract, the 2D MIN h R. ASSIGN. problem consists of finding a minimum cost range assignment for a given set S of radio stations on the plane provided that the assignment ensure the communication between any pair of stations in at most h hops, where h is an input integer parameter $(1 \leq h \leq |S| - 1)$.

1.1 Previous Works

Combinatorial optimization problems arising from the design of radio networks have been the subject of several papers over the last years (see [6] for a survey). In particular, NP-completeness results and approximation algorithm for scheduling communication in radio networks have been derived in [1,3,7,8]. Kirousis *et al*, in [4], investigated the complexity of the MIN R. ASSIGN. problem that consists of minimizing the overall transmission power assigned to a set S of stations of a radio network, provided that (multi-hop) communication is guaranteed for any pair of stations (notice that no bounds are required on the maximum number of hops for the communication). It turns out that the complexity of this problem depends on the dimension of the space the stations are located on. In the 1-dimensional case (i.e. when the stations are located along a line) they provide a polynomial-time algorithm that finds a range assignment of minimum cost. As for stations located in the 3-dimensional space they show that MIN R. ASSIGN. is NP-hard. They also provide a polynomial-time 2-approximation algorithm that works for any dimension. Then, Clementi *et al* in [2] proved that the MIN R. ASSIGN. problem in three dimensions is APX-complete thus implying that it does not admit PTAS unless P = NP (see [5] for a formal definition of these concepts). They also prove that the MIN R. ASSIGN. problem is NP-hard in the 2-dimensional case.

All the results mentioned above concern the case in which no restriction on the maximum number h of hops required by the communications among stations is imposed: a range assignment is feasible if it just guarantees a strong connectivity of the network. When, instead, a fixed bound on the number h of hops is imposed, the computational complexity of the corresponding problem is unknown (from [4,2] we know only that the problem is NP-hard for spaces of dimension at least 2 and $h = \Omega(n)$). However, in [4], two tight bounds for the minimum power consumption required by n points arranged on a unit chain are given (notice that in this case, given any n, there is only one instance of the problem).

Theorem 1 (The Unit Chain Case [4]). *Let N be a set of n colinear points at unit distance. Then the order of magnitude of the overall power required by any optimal range assignment of diameter h for N is respectively:*

- $\Theta\left(n^{\frac{2^{h+1}-1}{2^h-1}}\right)$, *for any fixed positive integer h;*
- $\Theta\left(\frac{n^2}{h}\right)$, *for any $h = \Omega(\log n)$.*

Furthermore the two above (implicit) upper bounds are constructive.

1.2 Our Results

We investigate the 2D MIN h R. ASSIGN. problem for constant values of h (i.e. when h is independent from the number of stations). We first provide the following general lower bound on the cost of optimal solutions for this problem.

Theorem 2. *For any set S of stations on the plane, let $\delta(S)$ be the minimum distance between any pair of different stations in S, and let $\mathsf{opt}_h(S)$ be the cost of an optimal range assignment. Then, it holds*

$$\mathsf{opt}_h(S) = \Omega(\delta(S)^2|S|^{1+1/h}),$$

for any fixed positive integer h.

The second result of this paper is an efficient method to derive a solution for any instance of our problem for fixed values of h. Given a set of stations S, let us define

$$D(S) = \max\{d(s_i, s_j) \mid s_i, s_j \in S\}.$$

Then, our method yields the following result.

Theorem 3. *For any set of stations S on the plane, it is possible to construct in time $O(h|S|)$ a feasible range assignment $r_h(S)$ such that*

$$\mathsf{cost}(r_h(S)) = \mathrm{O}(D(S)^2|S|^{1/h}),$$

for any fixed positive integer h.

The above bounds provide a fast evaluation of the order of magnitude of the power consumption required by (and sufficient to) any radio network on the plane. This may result useful in network design phase in order to efficiently select a good configuration. Indeed, instead of blindly trying and evaluating a huge number of tentative configurations, an easy application of our bounds could allow us to determine whether or not all such configurations are equivalent from the power consumption point of view.

Let us now consider the instance G_n of 2D MIN h R. ASSIGN. in which n stations are placed on a square grid of side $\sqrt{n}$ and the distance between adjacent pairs of stations is 1 (notice that this is the 2-dimensional version of the unit chain case studied in [4] - see Theorem 1).

Since our lower bound holds *for any* station set S, by combining Theorem 2 and 3, we easily obtain that

$$\mathsf{opt}_h(G_n) = \Theta\left(n^{1+1/h}\right). \tag{2}$$

The square grid configuration is the most regular case of *well-spread* instances. In general, we say that a family $\mathcal{S}$ of *well-spread* instances is a family of instances S such that $D(S) = O(\delta(S)\sqrt{|S|})$. Notice that the above property is rather natural: informally speaking, in a well-spread instance, any two stations must be not "too close". This is the typical situation in most of radio network configurations adopted in practice [6]. It turns out that the optimal bound in Eq. 2 holds for any family of well-spread instances. The following two corollaries are thus easy consequences of Theorems 2 and 3.

Corollary 1. *Let $\mathcal{S}$ be a family of well-spread instances. For any $S \in \mathcal{S}$, it holds that*

$$\mathsf{opt}_h(S) = \Theta\left(\delta(S)^2|S|^{1+1/h}\right),$$

for any positive integer constant h.

Corollary 2. *Let $\mathcal{S}$ be any family of well-spread instances. Then, for any positive integer constant h, the* 2D MIN h R. ASSIGN. *problem restricted to $\mathcal{S}$ admits a polynomial-time approximation algorithm with constant performance ratio (i.e. the restriction is in* APX*).*

2 Preliminaries

Let $S = \{s_1, \ldots, s_n\}$ be a set of n points (representing stations) in $\mathcal{R}^2$ with the Euclidean distance $d : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}^+$, where $\mathcal{R}^+$ denotes the set of non negative reals. We define

$$\delta(S) = \min\{d(s_i, s_j) \mid s_i, s_j \in S, i \neq j\}$$

and

$$D(S) = \max\{d(s_i, s_j) \mid s_i, s_j \in S\}.$$

A *range assignment* for S is a function $r : S \rightarrow \mathcal{R}^{+}$. The *cost* $\mathsf{cost}(r)$ of r is defined as

$$\mathsf{cost}(r) = \sum_{i=1}^{n} (r(s_i))^2.$$

Observe that we have set the distance-power gradient β to 2 (see Eq. 1), however our results can be easily extended to any constant $\beta \geq 1$.

The *communication graph* of a range assignment r is the directed graph $G_r(S, E)$ where $(s_i, s_j) \in E$ if and only if $r(s_i) \geq d(s_i, s_j)$. We say that an assignment r for S is of diameter h $(1 \leq h \leq n-1)$ if the corresponding communication graph is strongly connected and has diameter h (in short, an *h-assignment*).

As defined in the Introduction, given a set S of n points in $\mathcal{R}^2$ and a positive integer h, the 2D Min h R. Assign. problem consists of finding an h-assignment r_{min} for S of minimum cost. The cost of an optimal h-assignment for a given set of stations S is denoted as $\mathsf{opt}_h(S)$.

In the proof of our results we will make use of the well-known Hölder inequality. We thus present it in the following convenient form. Let x_i, $i = 1, \ldots, k$ be a set of k non negative reals and let $p, q \in \mathcal{R}$ such that $p \geq 1$ and $q \leq 1$. Then, it holds that:

$$\sum_{i=1}^{k} x_i^p \geq k \left(\frac{\sum_{i=1}^{k} x_i}{k} \right)^p; \tag{3}$$

$$\sum_{i=1}^{k} x_i^q \leq k \left(\frac{\sum_{i=1}^{k} x_i}{k} \right)^q. \tag{4}$$

3 The Lower Bound

Given a set S of stations and a "base" station $b \in S$, we define $\mathsf{opt}_h(S, b)$ as the minimum cost of any range assignment ensuring that any station $s \in S$ can reach b in at most h hops. By the definition of the 2D Min h R. Assign. problem, it should be clear that the cost required by any instance S of this problem is at least $\mathsf{opt}_h(S, b)$, for any $b \in S$. So, the main result of this section is an easy consequence of the following lemma.

Lemma 1. *Let S be any set of stations such that $\delta(S) = 1$. For every $b \in S$ and every positive constant integer h, it holds that*

$$\mathsf{opt}_h(S, b) = \Omega(|S|^{1+1/h}).$$

Proof. We first observe that, since $\delta(S) = 1$, for sufficiently large sets S (more precisely, for any S such that $|S| > 16$), the maximum number of stations contained in a disk of radius $R = \sqrt{|S|/3}$ is at most $|S|/2$.

Let $r_h^{all-to-one}$ be a range assignment that ensures that all the stations in S can reach b in at most h hops. We prove that $\mathsf{cost}(r_h^{all-to-one}) = \Omega(|S|^{1+1/h})$ by induction on h.
For $h = 1$, consider the disk of radius R and centered in b. By the above observation, there are at least $|S|/2$ stations at distance greater than R from b. The cost required by such stations to reach b in one hop is at least

$$(|S|/2)R^2 = \Omega(|S|^2).$$

Let $h \geq 2$, we define

$$FAR = \{s \in S \mid d(s, b) > R\}.$$

Clearly, we have that $|FAR| \geq |S|/2$. Every station s in FAR must reach b in $k \leq h$ hops, it thus follows that there exist $k \leq h$ positive reals $x_1, \ldots, x_k$ (where x_i is the distance covered by the i-th hop of the communication from s to b) such that

$$x_1 + x_2 + \cdots + x_k \geq R.$$

So, at least one index j exists for which $x_j \geq R/k \geq R/h$. We can thus define the set of "bridge" stations

$$B = \{s \in S \mid r_h^{all-to-one}(s) \geq R/h)\}.$$

Two cases may arise.

Case $|B| \geq |S|^{\frac{1}{h}}$. In this case, since $|R| = \sqrt{|S|/3}$,

$$\begin{aligned} \sum_{s \in S} (r_h^{all-to-one}(s))^2 &\geq |B|(R/h)^2 \\ &\geq \frac{1}{3h^2}|S|^{1+\frac{1}{h}} \\ &= \Omega\left(|S|^{1+\frac{1}{h}}\right). \end{aligned}$$

Case $|B| < |S|^{\frac{1}{h}}$. By means of the assignment $r_h^{all-to-one}$, every station in FAR reaches in at most $h-1$ hops some bridge station. Let $B = \{b_1, \ldots, b_{|B|}\}$. So, we can partition the set $FAR \cup B$ into $|B|$ subsets $A_1, \ldots, A_{|B|}$ such that all the stations in A_i reach b_i in at most $h-1$ hops[1]. So,

$$\begin{aligned} \sum_{s \in S} (r_h^{all-to-one}(s))^2 &\geq \sum_{i=1}^{|B|} \mathsf{opt}_{h-1}(A_i, b_i) \\ &= \Omega\left(\sum_{i=1}^{|B|} |A_i|^{1+\frac{1}{h-1}}\right) \end{aligned}$$

[1] Notice that if a station reaches two or more bridge stations, we can put the station into any of the corresponding set A_i's. We also assume that $b_i \in A_i$, for $1 \leq i \leq |B|$.

where the last bound is a consequence of the inductive hypothesis. Since

$$\sum_{i=1}^{|B|} |A_i| = |FAR \cup B| \geq |S|/2,$$

the Hölder inequality (see Eq. 3) implies that

$$\begin{aligned}\sum_{i=1}^{|B|} |A_i|^{1+\frac{1}{h-1}} &\geq |B| \left(\frac{|S|/2}{|B|}\right)^{1+\frac{1}{h-1}} \\ &= \Omega\left(\left(\frac{1}{|B|}\right)^{\frac{1}{h-1}} |S|^{1+\frac{1}{h-1}}\right) \\ &= \Omega\left(|S|^{1+\frac{1}{h}}\right)\end{aligned}$$

where the last equivalence is due to the condition $|B| < |S|^{\frac{1}{h}}$.

Proof of Theorem 2.
For $\delta(S) = 1$, the theorem is an immediate consequence of Lemma 1. The general case $\delta(S) > 0$ can be reduced to the previous case by simply rescaling the instance by a factor of $1/\delta(S)$.

□

4 The Upper Bound

Proof of Theorem 3.
The proof consists of a recursive construction of an h-assignment $r_h(S)$ having cost $O(D(S)^2|S|^{1/h})$. For $h = 1$, $r_1(S)$ assigns a range $D(S)$ to each station in S. Thus, $\mathsf{cost}(r_1(S)) = D(S)^2|S|$.

Let us consider the smallest square Q that contains all points in S. Notice that the side l of Q is at most $D(S)$. Let us consider a grid that subdivides Q into k^2 subsquares of the same size l/k (the choice of k will be given later).

Informally speaking, for every non empty subsquare we choose a “base” station and we give power sufficient to let it cover all the stations in S in one hop. Then, in every subsquare we complete the assignment by making any station able to reach the base station in $h-1$ hops. For this task we apply the recursive construction.

The cost of $r_h(S)$ is thus bounded by

$$\mathsf{cost}(r_h(S)) \leq k^2 D(S)^2 + \sum_{i=1}^{k^2} \mathsf{cost}(r_{h-1}(S_i)),$$

where S_i is the set of the stations in the i-th subsquare. Since $D(S_i) = O(D(S)/k)$ we apply the inductive hypothesis and we obtain

$$\begin{aligned}\mathsf{cost}(r_h(S)) &= \mathrm{O}\left(k^2 D(S)^2 + \sum_{i=1}^{k^2} |S_i|^{1/(h-1)} \left(\frac{D(S)}{k}\right)^2\right) \\ &= \mathrm{O}\left(k^2 D(S)^2 + \left(\frac{D(S)}{k}\right)^2 \sum_{i=1}^{k^2} |S_i|^{1/(h-1)}\right) \\ &= \mathrm{O}\left(k^2 D(S)^2 + \left(\frac{D(S)}{k}\right)^2 k^2 \left(\frac{|S|}{k^2}\right)^{1/(h-1)}\right),\end{aligned}$$

where the last equality follows from the Hölder inequality (see Eq. 4) and from the fact that $\sum_{i=1}^{k^2} |S_i| = |S|$. Now we choose

$$k = |S|^{\frac{1}{2h}}$$

in order to equate the additive terms in the last part of the above equation. By replacing this value in the equation we obtain

$$\mathsf{cost}(r_h(S)) = \mathrm{O}\left(D(S)^2 |S|^{1/h}\right).$$

It is easy to verify that the partition of Q into k^2 subsquares and the rest of the computation in each inductive step can be done in time $O(|S|)$. So, the overall time complexity is $O(h|S|)$.

□

5 Tight Bounds and Approximability

Let us consider the simple instance G_n of 2D Min h R. Assign. in which n stations are placed on a square grid of side $\sqrt{n}$, and the distance between adjacent pairs of stations is 1.

By Combining Theorems 2 and 3, we easily obtain that

$$\mathsf{opt}_h(G_n) = \Theta\left(n^{1+1/h}\right).$$

This also implies that the range assignment constructed in the proof of Theorem 3 yields a constant-factor approximation.

It turns out that the above considerations can be extended to any "well-spread" instance.

Definition 1. *A family $\mathcal{S}$ of* well-spread *instances is a family of instances S such that $D(S) = O(\delta(S)\sqrt{|S|})$.*

The following two corollaries are easy consequences of Theorems 2 and 3.

Corollary 3. *Let $\mathcal{S}$ be a family of well-spread instances. Then, For any $S \in \mathcal{S}$,*

$$\mathsf{opt}_h(S) = \Theta\left(\delta(S)^2 |S|^{1+1/h}\right),$$

for any positive integer constant h.

Corollary 4. *Let $\mathcal{S}$ be any family of well-spread instances.*
Then, the 2D MIN h R. ASSIGN. *problem restricted to $\mathcal{S}$ is in* APX, *for any positive integer constant h.*

6 Open Problems

As discussed in the Introduction, finding bounds for the power consumption of general classes of radio networks might result very useful in network design. Thus it is interesting to derive new, tighter bounds for possibly a larger class of radio network configurations.
Another question left open by this paper is whether the 2D MIN h R. ASSIGN. is NP-hard for constant h. We conjecture a positive answer even though we believe that any proof will depart significantly from those adopted for the unbounded cases (i.e. the MIN R. ASSIGN. problem - see [4,2]). More precisely, all the reductions adopted in the unbounded cases start from the minimum vertex cover problem that seems to be very unsuitable for our problem. When the stations are arranged on a line (i.e. the 1-dimensional case), we instead conjecture that the problem is in P for any value of h.

Acknowledgments Part of this work has been done while the authors were visiting the research center of INRIA Sophia Antipolis, partially supported by the INRIA project COMMOBIL.

References

1. E. Arikan, “Some Complexity Results about Packet Radio Networks”, IEEE Transactions on Information Theory, 456-461, IT-30, 1984.
2. A.E.F. Clementi, P. Penna, and R. Silvestri, “Hardness Results for the Power Range Assignment Problem in Packet Radio Networks”, II International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX’99), LNCS, 1671, 1999.
3. A. Ephemides and T. Truong, “Scheduling Broadcast in Multihop Radio Networks”, IEEE Transactions on Communications, 456-461, 30, 1990.
4. L. M. Kirousis and E. Kranakis and D. Krizanc and A. Pelc, “Power Consumption in Packet Radio Networks”, 14th Annual Symposium on Theoretical Aspects of Computer Science (STACS 97), LNCS 1200, 363-374, 1997 (to appear also in TCS).
5. C.H. Papadimitriou, “Computational Complexity”, Addison-Wesley Publishing Company, Inc., 1995.

6. K. Pahlavan and A. Levesque, "Wireless Information Networks", Wiley-Interscience, New York, 1995.
7. S. Ramanathan and E. Lloyd, "Scheduling Broadcasts in Multi-hop Radio Networks" IEEE/ACM Transactions on Networking, 166-172, 1, 1993.
8. R. Ramaswami and K. Parhi, "Distributed Scheduling of Broadcasts in Radio Network", INFOCOM'89, 497-504, 1989.