Hardness Results for the Power Range Assignment Problem in Packet Radio Networks

(Extended Abstract)

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Abstract. The *minimum range assignment problem* consists of assigning transmission ranges to the stations of a multi-hop packet radio network so as to minimize the total power consumption provided that the transmission range assigned to the stations ensures the strong connectivity of the network (i.e. each station can communicate with any other station by multi-hop transmission). The complexity of this optimization problem was studied by Kirousis, Kranakis, Krizanc, and Pelc (1997). In particular, they proved that, when the stations are located in a 3-dimensional Euclidean space, the problem is NP-hard and admits a 2-approximation algorithm. On the other hand, they left the complexity of the 2-dimensional case as an open problem.
As for the 3-dimensional case, we strengthen their negative result by showing that the minimum range assignment problem is APX-complete, so, it does not admit a polynomial-time approximation scheme unless P = NP.
We also solve the open problem discussed by Kirousis *et al* by proving that the 2-dimensional case remains NP-hard.

1 Introduction

A *Multi-Hop Packet Radio Network* [10] is a set of radio stations located on a geographical region that are able to communicate by transmitting and receiving radio signals. A transmission range is assigned to each station s and any other station t within this range can directly (i.e. by one *hop*) receive messages from s. Communication between two stations that are not within their respective ranges can be achieved by *multi-hop* transmissions. In general, Multi-Hop Packet Radio Networks are adopted whenever the construction of more traditional networks is impossible or, simply, too expensive.

It is reasonably assumed [10] that the power P_t required by a station t to correctly transmit data to another station s must satisfy the inequality

$$\frac{P_t}{d(t,s)^{\beta}} > \gamma \tag{1}$$

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where $d(t, s)$ is the distance between t and s, $\beta \geq 1$ is the *distance-power gradient*, and $\gamma \geq 1$ is the *transmission-quality* parameter. In an ideal environment (see [10]) $\beta = 2$ but it may vary from 1 to 6 depending on the environment conditions of the place the network is located. In the rest of the paper, we fix $\beta = 2$ and $\gamma = 1$, however, our results can be easily extended to any $\beta, \gamma > 1$.

Combinatorial optimization problems arising from the design of radio networks have been the subject of several papers over the last years (see [10] for a survey). In particular, NP-completeness results and approximation algorithm for scheduling communication and power range assignment problems in radio networks have been derived in [2, 6, 13, 14].

More recently, Kirousis *et al*, in [9], investigated the complexity of the MIN RANGE ASSIGNMENT problem that consists of minimizing the overall transmission power assigned to the stations of a radio network, provided that (multi-hop) communication is guaranteed for any pair of stations (for a formal definition see Section 2). It turns out that the complexity of this problem depends on the number of dimensions of the space the stations are located on. In the 1-dimensional case (i.e. when the stations are located along a line) they provide a polynomial-time algorithm that finds a range assignment of minimum cost. As for stations located in the 3-dimensional space, they instead derive a polynomial-time reduction from MIN VERTEX COVER restricted to planar cubic[1] graphs thus showing that MIN RANGE ASSIGNMENT is NP-hard. They also provide a polynomial-time 2-approximation algorithm that works for any dimension.

In this paper, we address the question whether the approximation algorithm given by Kirousis *et al* for the MIN RANGE ASSIGNMENT problem in three dimensions can be significantly improved. More precisely, we ask whether or not the problem does admit a *Polynomial-Time Approximation Scheme* (PTAS). We indeed demonstrate the APX-completeness of this problem thus implying that it does not admit PTAS unless P = NP (see [12] for a formal definition of these concepts).

The standard method to derive an APX-completeness result for a given optimization problem Π is: *i*) consider a problem Π' which is APX-hard and then *ii*) show an *approximation-preserving* reduction from Π' to Π [12]. We emphasize that Kirousis *et al*'s reduction does not satisfy any of these two requirements. In fact, as mentioned above, their reduction is from MIN VERTEX COVER restricted to planar cubic graphs which cannot be APX-hard (unless P = NP) since it admits a PTAS [3]. Furthermore, it is not hard to verify that their reduction is not approximation-preserving.

In order to achieve our hardness result, we instead consider the MIN VERTEX COVER problem restricted to cubic graphs which is known to be APX-complete [11, 1] and then we show an approximation-preserving reduction from this variant of MIN VERTEX COVER to MIN RANGE ASSIGNMENT in three dimensions. Furthermore, our reduction is "efficient", we obtain an interesting explicit relationship between the approximability behaviour of MIN VERTEX COVER and that of the 2-dimensional MIN RANGE ASSIGNMENT problem.

[1] A graph is *cubic* when every node has degree 3.

In fact, we can state that if MIN VERTEX COVER on cubic graphs is not $\overline{\rho}$-approximable then MIN RANGE ASSIGNMENT in three dimensions is not $\frac{\overline{\rho}+4}{5}$-approximable.

Kirousis *et al*'s reduction works only in the 3-dimensional case. In fact, the reduction starts from a planar orthogonal drawing of a (planar) cubic graph G and replace each edge by a *gadget* of stations drawn in the 3-dimensional space that "simulates" the connection between the two adjacent nodes. In order to preserve pairwise "independence" of the drawing of gadgets, their reduction strongly uses the third dimension left "free" by the planar drawing of G. The complexity of the MIN RANGE ASSIGNMENT problem in two dimensions is thus left as an open question: Kirousis *et al* in fact conjectured the NP-hardness of this restriction.

It turns out that the gadget construction used in our approximation-preserving reduction for the 3-dimensional case can be suitably adapted in order to derive a polynomial-time reduction from MIN VERTEX COVER on planar cubic graphs to the 2-dimensional MIN RANGE ASSIGNMENT problem thus proving their conjecture. The following table summarizes the results obtained in this paper.

Problem version	**Previous results**	**Our results**
1-Dim. Case	in P[9]	-
2-Dim. Case	in APX[9]	NP-complete
3-Dim. Case	NP-complete, in APX[9]	APX-complete

Organization of the Paper. In Section 2, we give the preliminary definitions. For the sake of convenience, we first provide the reduction proving the NP-completeness result for the 2-dimensional case in Section 3. Then, in Section 4, we show the APX-completeness of MIN RANGE ASSIGNMENT in the 3-dimensional case. Finally, some open problems are discussed in Section 5. The proofs of the technical lemmas will be given in the full version of the paper.

2 Preliminaries

Let $S = \{s_1, \ldots, s_n\}$ be a set of n points (representing stations) of an Euclidean space $\mathcal{E}$ with distance function $d : \mathcal{E}^2 \rightarrow \mathcal{R}^+$, where $\mathcal{R}^+$ denotes the set of non negative reals. A *range assignment* for S is a function $r : S \rightarrow \mathcal{R}^+$. The *cost* $\mathsf{cost}(r)$ of r is defined as

$$\mathsf{cost}(r) \; = \; \sum_{i=1}^{n} (r(s_i))^2 \; .$$

Observe that we have set the distance-power gradient β to 2 (see Eq. 1), however our results can be easily extended to any constant $\beta > 1$.

The *communication graph* of a range assignment r is the directed graph $G_r(S, E)$ where $(s_i, s_j) \in E$ if and only if $r(s_i) \geq d(s_i, s_j)$. We say that an assignment r for S is *feasible* if the corresponding communication graph is strongly connected. Given a set S of n points in an Euclidean space, the Min Range Assignment problem consists of finding a feasible range assignment r_{min} for S of minimum cost. With 2D Min Range Assignment (respectively, 3D Min Range Assignment) we denote the Min Range Assignment problem in which the points are placed on $\mathcal{R}^2$ (respectively, on $\mathcal{R}^3$).

The Min Vertex Cover problem is to find a subset K of the set of vertices of V of a graph $G(V, E)$ such that K contains at least one endpoint of any edge in E and $|K|$ is as small as possible. Min Vertex Cover is known to be NP-hard even when restricted to planar cubic graphs [7]. Moreover, it is known to be APX-complete when restricted to cubic graphs [11, 1]. It follows that a constant $\overline{\rho} > 1$ exists such that Min Vertex Cover restricted to cubic graphs is not $\overline{\rho}$-approximable unless P = NP.

3 2D Min Range Assignment is NP-hard

We will show a polynomial-time reduction from Min Vertex Cover restricted to planar, cubic graphs to 2D Min Range Assignment.

Given a planar, cubic graph $G(V, E)$, it is always possible to derive a planar orthogonal drawing of G in which each edge is represented by a polyline having only one bend [15, 8]. We can then replace every edge whose drawing has one bend with a chain of three edges (we add two new vertices) in such a way that all edges are represented by straightline segments. The obtained drawing will be denoted by $D(G)$. It is easy to verify that, if $2h$ is the number of vertices added by this operation, then G has a vertex cover of size k if and only if $D(G)$ has a vertex cover[2] of size $k + h$. As we will see in Subsection 3.2, further vertices will be added in $D(G)$ still preserving the above relationship between the vertex covers of G and those of $D(G)$.

Our goal is to replace each edge (and thus both of its vertices) of $D(G)$ with a gadget of points (stations) in the Euclidean space $\mathcal{R}^2$ in order to construct an instance of the 2D Min Range Assignment problem and then show that this construction is a polynomial-time reduction. In the next subsection we provide the key properties of these gadgets and the reduction to 2D Min Range Assignment that relies on such properties. The formal construction of the 2-dimensional gadgets is instead given in Subsection 3.2.

3.1 The Properties of the 2-Dimensional Gadgets and the Reduction

The type of gadget used to replace one edge of $D(G)$ depends on the local "situation" that occurs in the drawing (for example it depends on the degree of its endpoints). However, we can state the properties that characterize any of these gadgets.

[2] In what follows, we will improperly $D(G)$ to denote both the drawing and the graph it represents.

Definition 1 (Gadget Properties). *Let $\delta, \delta', \epsilon \geq 0$ such that $\delta + \epsilon > \delta'$ and $\alpha > 1$ (a suitable choice of such parameters will be given later). For any edge (a, b) the corresponding gadget g_{ab} contains the sets of points $X_{ab} = \{x_1, \ldots, x_{l_1}\}$, $Y_{ab} = \{y_{ab}, y_{ba}\}$, $Z_{ab} = \{z_1, \ldots, z_{l_2}\}$ and $V_{ab} = \{a, b\}$, where l_1 and l_2 depend on the length of the drawing of (a, b). These sets of points are drawn in $\mathcal{R}^2$ so that the following properties hold:*

1. *$d(a, y_{ab}) = d(b, y_{ba}) = \delta + \epsilon$.*
2. *X_{ab} is a chain of points drawn so that $d(a, x_1) = \delta$ and $d(b, x_{l_1}) = \delta$. Furthermore, for any $i = 1, \ldots, l_1 - 1$, $d(x_i, x_{i+1}) = \delta$ and, for any $i \neq j$, $d(x_i, x_j) \geq \delta$.*
3. *Z_{ab} is a chain of points drawn so that $d(y_{ab}, z_1) = d(y_{ba}, z_{l_2}) = \delta'$. Furthermore, for any $i = 1, \ldots, l_2 - 1$, $d(z_i, z_{i+1}) = \delta'$ and, for any $i \neq j$, $d(z_i, z_j) \geq \delta'$.*
4. *For any $x_i \in X_{ab}$ and $z_j \in Z_{ab}$, $d(x_i, z_j) > \delta + \epsilon$. Furthermore, for any $i = 1, \ldots, l_1$, $d(x_i, y_{ab}) \geq \delta + \epsilon$ and $d(x_i, y_{ba}) \geq \delta + \epsilon$.*
5. *Given any two different gadgets g_{ab} and g_{cd}, for any $v \in g_{ab} \setminus g_{cd}$ and $w \in g_{cd} \setminus g_{ab}$, we have that $d(v, w) \geq \delta$ and if $v \notin V_{ab} \cup X_{ab}$ or $w \notin V_{cd} \cup X_{cd}$ then $d(v, w) \geq \alpha\delta$.*

From the above definition, it turns out that the gadgets consist of two components whose relative distance is $\delta + \epsilon$: the VX*-component* consisting of the "chain" of points in $X_{ab} \cup V_{ab}$, and the YZ*-component* consisting of the chain of points in $Y_{ab} \cup Z_{ab}$.

Let $S(G)$ be the set of points obtained by replacing each edge of $D(G)$ by one gadget having the properties described above.

Note 1. Let r^{min} be the range assignment of $S(G)$ in which every point in VX and in YZ have range δ and δ', respectively (notice that this assignment is not feasible). The corresponding communication graph consists of $m + 1$ strongly connected components, where m is the number of edges: the YZ-components of the m gadgets and the union $\mathcal{U}$ of all the VX-components of the gadgets. It thus follows that, in order to achieve a feasible assignment, we must define the "bridge-point" between $\mathcal{U}$ and every YZ-component.

The above note leads us to define the following *canonical* (feasible) solutions for $S(G)$.

Definition 2 (Canonical Solutions for $S(G)$). *A range assignment r for $S(G)$ is* canonical *if, for every gadget g_{ab} of $S(G)$, the following properties hold.*

1. *Either $r(y_{ab}) = \delta + \epsilon$ and $r(y_{ba}) = \delta'$ (so, y_{ab} is a radio "bridge" from the YZ-component to the VX one) or vice versa.*
2. *For every $v \in \{a, b\}$, either $r(v) = \delta$ or $r(v) = \delta + \epsilon$. Furthermore, there exists $v \in \{a, b\}$ such that $r(v) = \delta + \epsilon$ (so, v is a radio "bridge" from the VX-component to the YZ one).*
3. *For every $x \in X_{ab}$, $r(x) = \delta$.*

4. For every $z \in Z_{ab}$, $r(z) = \delta'$.

We observe that any canonical assignment is feasible.

Lemma 1. *Let us consider the construction $S(G)$ in which α, δ and ϵ are three positive constants such that*

$$\alpha^2\delta^2 \; > \; (m-1)[(\delta+\epsilon)^2 - \delta^2] + (\delta+\epsilon)^2. \tag{2}$$

Then, for any feasible range assignment r for $S(G)$, there is a canonical range assignment r^c such that $\mathsf{cost}(r^c) \le \mathsf{cost}(r)$.

We now assume that $S(G)$ satisfies the hypothesis of Lemma 1.

Lemma 2. *Given any planar cubic graph $G(V,E)$, assume that it is possible to construct the set of points $S(G)$ in the plane in time polynomial in the size of G. Then* Min Vertex Cover *is polynomial-time reducible to* 2d Min Range Assignment.

3.2 The Construction of the 2-Dimensional Gadgets

This section is devoted to the construction of the 2-dimensional gadgets that allow us to obtain the point set $S(G)$ corresponding to a given planar cubic graph G.

Definition 3 (Construction of $S(G)$). *Let $G(V,E)$ be a planar cubic graph, then the set of points $S(G)$ is constructed as follows:*

1. *Construct a planar orthogonal grid drawing of G with at most one bend per edge.*
2. *For any edge represented by a polyline with one bend, add two new vertices so that any edge is represented with a straight line segment.*
3. *Starting from the obtained graph $D(G)$, replace its edges with the gadgets satisfying Definition 1 and Eq. 2. This step may require further vertices to be added to $D(G)$ while preserving the relationship between the vertex cover solutions.*

Let us first observe that G has a vertex cover of size k if and only if $D(G)$ has a vertex cover of size $k+h$, where $2h$ is the number of new vertices added in the last two steps. As we will see in the sequel h is polynomially bounded in the size of G. We can therefore consider the problem of finding a minimum vertex cover for $D(G)$.

During the third step of the construction, it is required to preserve Property 5 of Definition 1, i.e., points from different gadgets are required to be within distance at least $\alpha\delta$. Informally speaking, the main technical problem is drawing the Z-chains corresponding to incident edges so that the properties of Definition 1 hold. To this aim, we adopt a set of suitable construction rules that are described in the full version of the paper.

In the sequel the term $S(G)$ will denote the network drawn from $D(G)$ according to the construction rules mentioned above. Let L_{min} be the minimum distance between any two V-points in $D(G)$. Then, any two V-points of the obtained network $S(G)$ have distance not smaller than L_{min}.

Lemma 3. *Let $\delta = L_{min}/6$. Then, an $\epsilon > 0$ exists for which the corresponding network $S(G)$ satisfies Eq. 2, i.e.,*

$$\alpha^2\delta^2 \;>\; (m-1)[(\delta+\epsilon)^2-\delta^2]+(\delta+\epsilon)^2$$

where

$$\alpha = \frac{1+\sqrt{2}}{2}\ .$$

Combining Lemma 2 with Lemma 3 we obtain the following result.

Theorem 1. 2D Min Range Assignment *is* NP-*hard.*

4 3D Min Range Assignment is APX-complete

The APX-completeness of 3D Min Range Assignment is achieved by showing an approximation-preserving reduction from Min Vertex Cover restricted to cubic graphs, a restriction of Min Vertex Cover which is known to be APX-complete [11, 1]. The approximation-preserving reduction follows the same idea of the reduction shown in the previous section and thus requires a suitable 3-dimensional drawing of a cubic graph.

Theorem 2. *[5] There is a polynomial-time algorithm that, given any cubic graph $G(V, E)$, returns a 3-dimensional orthogonal drawing $D(G)$ of G such that:*

- *Every edge is represented as a polyline with at most three bends.*
- *Vertices are represented as points with integer coordinates, thus the minimum distance L_{min} between two vertices is at least 1.*
- *The maximum length L_{max} of an edge in $D(G)$ is polynomially bounded in $m = |E|$.*

4.1 The 3-Dimensional Gadgets

In what follows, we assume to have at hand the 3-dimensional, orthogonal drawing $D(G)$ of a cubic graph G that satisfies the properties of Theorem 2. Then the approximation-preserving reduction replaces each edge of $D(G)$ with a 3-dimensional gadget of stations having the following properties.

Definition 4 (Properties of 3-Dimensional Gadgets).
Let l and ϵ be positive constants (a suitable choice of such parameters will be given later). For any edge (a, b) the corresponding gadget contains the sets of points $X_{ab} = \{x_1, \ldots, x_{l_1}\}$, $Y_{ab} = \{y_{ab}, y_{ba}\}$, $Z_{ab} = \{z_1, \ldots, z_{l_2}\}$ and $V_{ab} = \{a, b\}$, where l_1 and l_2 depend on the distance $d(a, b)$ and $d(y_{ab}, y_{ba})$, respectively. The above set of points is drawn in such a way that the following properties hold:

1. *$d(a, y_{ab}) = d(b, y_{ba}) = l$.*
2. *X_{ab} and Z_{ab} are two chains of points drawn so that $d(a, x_1) = d(b, x_l) = \epsilon$ and $d(y_{ab}, z_1) = d(y_{ba}, z_m) = \epsilon$, respectively. Furthermore, for any $i = 1, \ldots, l-1$, $d(x_i, x_{i+1}) = \epsilon$ and for any $j = 1, \ldots, m-1$ $d(z_j, z_{j+1}) = \epsilon$.*
3. *For any $x_i \in X_{ab}$ and $z_j \in Z_{ab}$, $d(x_i, z_j) > l$. Furthermore $d(x_i, y_a b) \geq l$ and $d(x_i, y_b a) \geq l$.*
4. *Given any two different gadgets g_1 and g_2, for any $v \in g_1$ and $w \in g_2$ with $u \neq w$ of different type (for example, if u is a X-point then w is either a Y-point or a Z-point), we have that $d(v, w) > l$. Moreover, the minimum distance between the YZ-component[3] of g_1 and the YZ-component of g_2 is $2l$.*
5. *Given any two* non adjacent *gadgets g_1 and g_2, for any $v \in g_1$ and $w \in g_2$, $d(v, w) \geq L_{min}/2$.*

Let l and ϵ two positive reals such that $l \leq L_{min}$ (this assumption guarantees Properties 4 and 5 of Definition 4) and $\epsilon < l$. The construction of the 3-dimensional gadgets can be obtained by adopting the same method of the 2-dimensional case. The technical differences will be discussed in the full version of the paper.

We emphasize that the 3-dimensional gadgets have two further properties which will be strongly used to achieve an approximation-preserving reduction (see Theorem 3).

Lemma 4. 1). *The set of V-points of $S(G)$ is the set of vertices of G, i.e. no new vertices will be added with respect to those of $D(G)$.*

2). *It is possible to make the overall range cost of both X and Z points of any gadget arbitrarily small by augmenting the number of equally spaced stations in these two chains. More formally, if L is the length of the polyline representing an edge (a, b) in $D(G)$ and k is the number of points in the X (or Z) component then the overall power needed for the X component is*

$$(k+2)\left(\frac{L}{k+1}\right)^2 \tag{3}$$

So, by increasing k, we can make the above value smaller than any fixed positive constant.

4.2 The Approximation-Preserving Reduction

Definition 5 (Canonical Solutions for $S(G)$.). *A range assignment r for $S(G)$ is* canonical *if, for every gadget g_{ab} of $S(G)$, the following properties hold.*

1. *Either $r(y_{ab}) = l$ and $r(y_{ba}) = \epsilon$ (so, y_{ab} is the radio "bridge" from the YZ-component to the VX one) or vice versa.*

[3] Similarly to the 2-dimensional case, the sets of points $V_{ab} \cup X_{ab}$ and $Y_{ab} \cup Z_{ab}$ will be denoted as VX-component and YZ-component, respectively.

2. *For every* $v \in \{a, b\}$, *either* $r(v) = \epsilon$ *or* $r(v) = l$. *Furthermore, there exists* $v \in \{a, b\}$ *such that* $r(v) = l$ *(so,* v *is a radio "bridge" from the* VX*-component to the* YZ *one).*
3. *For every* $x \in X_{ab}$, $r(x) = \epsilon$.
4. *For every* $z \in Z_{ab}$, $r(z) = \epsilon$.

Lemma 5. *For any graph* G, *let us consider the construction* $S(G)$ *in which* l *is a positive real that satisfies the following inequality*

$$l^2 < \frac{L_{min}^2}{m} . \tag{4}$$

Then, for any feasible range assignment r *of* $S(G)$, *there is a canonical range assignment* r^c *such that* $\mathsf{cost}(r^c) \leq \mathsf{cost}(r)$.

Informally speaking, the presence of the third dimension in placing the gadgets allows to keep a polynomially large gap between the value of l (i.e. the minimum distance between the VX component and the YX one of a gadget) and that of ϵ (i.e. the minimum distance between points in the same chain component). This gap yields the significant weight of each bridge-point of type V in a canonical solution and it will be a key ingredient in proving the next theorem. Notice also that this gap cannot be smaller than a fixed positive constant in the 2-dimensional reduction shown in the previous section.

Theorem 3. 3D Min Range Assignment *is* APX-*complete.*

Proof. The outline of the proof is the following. We assume that we have at hand a polynomial-time ρ-approximation algorithm $\mathcal{A}$ for 3D Min Range Assignment. Then, we show a polynomial-time method that transforms $\mathcal{A}$ into a ρ'-approximation algorithm for Min Vertex Cover on cubic graphs with $\rho' \leq 5\rho - 4$. Since a constant $\overline{\rho} > 1$ exists such that Min Vertex Cover restricted to cubic graphs is not $\overline{\rho}$-approximable unless P = NP [11, 1], the theorem follows.

Assume that a 3-degree graph $G(V, E)$ is given. Then, from the 3-dimensional orthogonal drawing $D(G)$ of G, we construct the radio network $S(G)$ by replacing each edge in $D(G)$ with one 3-dimensional gadget whose properties are described in Definition 4. It is possible to prove (see the full version of the paper) that these gadgets can be constructed and correctly placed in the 3-dimensional space in polynomial time. We also assume that the parameter l of $S(G)$ satisfies Inequality 4. Using the same arguments in the proof of Lemma 2, we can show that any vertex cover $K \subseteq V$ of G yields a canonical assignment r_K whose cost is

$$\mathsf{cost}(r_K) = \kappa l^2 + m l^2 + \overline{\epsilon_K}, \tag{5}$$

where $\kappa = |K|$ and $\overline{\epsilon_K}$ is the overall cost due to all points v that have range ϵ. Since each gadget of $S(G)$ has at most $4L_{max}/\epsilon$ points, it holds that

$$\overline{\epsilon_K} \leq 4 m L_{max} \epsilon . \tag{6}$$

On the other hand, from Lemma 5, we can consider only canonical solutions of $S(G)$. Thus, given a canonical solution r^c, we can consider the subset K of V-points whose range is l. It is easy to verify that K is a vertex cover of G. Furthermore, the cost of r^c can be written as follows

$$\mathsf{cost}(r^c) = |K|l^2 + ml^2 + \overline{\epsilon_K} \ .$$

Let K^{opt} be an optimum vertex cover for G, from the above equation we have that the optimum range assignment cost opt_r can be written as

$$\mathsf{opt}_r = |K^{opt}|l^2 + ml^2 + \overline{\epsilon_{K^{opt}}} \tag{7}$$

Since G has maximum degree 3 then $|K^{opt}| \geq m/3$; so, the above equation implies that

$$\mathsf{opt}_r \leq 4|K^{opt}|l^2 + \overline{\epsilon_{K^{opt}}}. \tag{8}$$

Let us now consider a ρ-approximation algorithm for 3D MIN RANGE ASSIGNMENT such that given $S(G)$ in input it returns a solution r^{apx} whose cost is less than $\rho \cdot \mathsf{opt}_r$. From Lemma 5, we can assume that r^{apx} is canonical. It thus follows that the cost $\mathsf{cost}(r^{apx})$ can be written as

$$\mathsf{cost}(r^{apx}) = |K^{apx}|l^2 + ml^2 + \overline{\epsilon_{K^{apx}}} \ .$$

From Eq.s 7 and 8 we obtain

$$\frac{\mathsf{cost}(r^{apx})}{\mathsf{opt}_r} = \frac{\mathsf{cost}(r^{apx}) - \mathsf{opt}_r}{\mathsf{opt}_r} + 1 \tag{9}$$

$$= \frac{|K^{apx}|l^2 + ml^2 + \overline{\epsilon_{K^{apx}}} - |K^{opt}|l^2 - ml^2 - \overline{\epsilon_{K^{opt}}}}{\mathsf{opt}_r} + 1 \tag{10}$$

$$\geq \frac{|K^{apx}|l^2 - |K^{opt}|l^2}{4|K^{opt}|l^2 + \overline{\epsilon_{K^{opt}}}} + 1 \tag{11}$$

Note that we can make $\overline{\epsilon_{K^{opt}}}$ arbitrarily small (independently from l) by reducing the parameter ϵ in the construction of $S(G)$: this is in turn obtained by increasing the number of X and Z points in the gadgets (see Lemma 4).

From Eq. 6, from the fact that L_{max} is polynomially bounded in the size of G and from the fact that l and L_{max} are polynomially related, we can ensure that $\overline{\epsilon_{K^{opt}}} \leq l^2$ by adding a polynomial number of points (see again Lemma 4). So, from Eq. 9 we obtain

$$\frac{\mathsf{cost}(r^{apx})}{\mathsf{opt}_r} \geq \frac{|K^{apx}|l^2 - |K^{opt}|l^2}{4|K^{opt}|l^2 + \overline{\epsilon_{K^{opt}}}} + 1 \geq \frac{|K^{apx}|}{5|K^{opt}|} + \frac{4}{5} \ .$$

Finally, it follows that the approximation ratio for MIN VERTEX COVER is bounded by

$$\frac{|K^{apx}|}{|K^{opt}|} \leq \frac{5\mathsf{cost}(r^{apx})}{\mathsf{opt}_r} - 4.$$

Q.E.D.

5 Open Problems

The first open problem is whether 2D Min Range Assignment is APX-complete or admits a PTAS. Notice that a possible APX-completeness reduction should be from a different problem, since Min Vertex Cover restricted to planar graphs is in PTAS. As regard the 3D Min Range Assignment problem it could be interesting to reduce the large gap between the factor 2 of the approximation algorithm and the inaproximability bound than can be derived by combining our reduction with the approximability lower bound of Min Vertex Cover on cubic graphs. As far as we know, there is no known significant explicit lower bound for the latter problem (an explicit 1.0029 lower bound for Min Vertex Cover on degree 5 graphs is given in [4] that – if it could be extended to cubic graphs and then combined with our reduction – would give a lower bound for 3D Min Range Assignment of 1.00059).

A crucial characteristic of the optimal solutions for the 3D Min Range Assignment instances given by our reduction is that stations that communicate directly have relative distance either l or ϵ, where $l >> \epsilon$. It could be interesting to consider instances in which the above situation does not occur. Notice that this is the case of the 2D Min Range Assignment instances of our reduction. Thus, the problem on such restricted instances remains NP-hard. However, it is an open problem whether a better approximation factor or even a PTAS can be obtained.

Another interesting aspect concerns the maximum number of hops required by any two stations to communicate. This corresponds to the diameter h of the communication graph. Our constructions yield solutions whose communication graph has unbounded (i.e. linear in the number of stations) diameter. So, the complexity of Min Range Assignment with bounded diameter remains open also in the 1-dimensional case. A special case where stations are placed at uniform distance on a line and either h is constant or $h \in O(\log n)$ has been solved in [9].

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